MODIFIED BOGOLYUBOV'S DERIVATION OF THE TWO-FLUID HYDRODYNAMICS

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A consistent microscopic derivation of the two-fluid hydrodynamics for superfluid helium-4 in the ideal approximation is represented. The starting point in our formalism is a system of Heisenberg's equation of motion for both normal and anomalous correlation functions. The use of a mixed Wigner representation allows us to perform the expansion of the equations of motion for correlation functions in gradients directly, very easily, and with a rigorous mathematics. To find the hydrodynamic flows, we have constructed a local equilibrium statistical operator for superfluid helium in the reference frame, where the condensate is at rest.

1. Introduction

Superfluid ^{4}He is a quantum degenerate system with spontaneously broken symmetry. Its feature is the macroscopic occupation of the lowest-energy single-particle quantum state in the momentum space or, in other words, is the presence of a condensate. As a result, the state of statistical equilibrium of the system with spontaneously broken symmetry depends on eight quantities: the particle density ρ, energy density ε, momentum density $\mathbf{j}$, and superfluid velocity $\mathbf{v}_s$. The presence of an additional velocity field leads to that the hydrodynamics of such a system is two-fluid.

The two-fluid hydrodynamic equations for superfluid ^{4}He in the phenomenological consideration were constructed by Landau in 1941 [1]. A semiphenomenological approach to the derivation of the two-fluid model in terms of the Boltzmann kinetic equation for elementary excitations was suggested in [2]. These equations were derived at the microscopic level by Bogolyubov in 1963 [3].

A starting point in the Bogolyubov's paper is the system of equations of motion for local quantities (particle density, momentum density, and energy density) which easy follows from the Heisenberg equations for both creation and annihilation operators, as well as the equation for the anomalous average $\langle\psi\rangle$ which yields the hydrodynamic equation for the superfluid velocity.

To pass from the formal equations of motion to hydrodynamic equations, Bogolyubov considered the stage of evolution when the system tends to equilibrium. Then it is possible to assume the establishment of a local equilibrium in the system. This is described by the statistical operator with parameters that depend on space coordinates. While approaching the thermodynamic equilibrium, these parameters are slowly changed in space and time. Therefore, their gradients are small. The procedure of expansion in the gradients is formulated by introducing the so-called "parameter of homogeneity" in the equations of motion. Then the expansion in the gradients coincides with the expansion in this parameter. We note that the introduction of the parameter of homogeneity was carried out in the Bogolyubov's paper in a formal way.

When the conservation relations for the local hydrodynamic quantities are constructed, the next step is the calculation of a hydrodynamic flows. Bogolyubov calculated the momentum flow by using a very elegant "scale transformation" method. But the flow of energy was obtained inconsistently. A more acceptable method of calculation of the energy flow with the use of an explicit local equilibrium statistical operator was proposed by Morozov [4].

Our paper imitates the Bogolyubov's article [3], but we work with the equations of motion for the correlation functions which are written in a mixed Wigner representation. The use of this representation allows us to make expansion of the exact equation of motion for correlation functions in the gradients without introduction of the Bogolyubov's "parameter of homogeneity". In other words, it allows us to realize the expansion in the gradients directly, easily, and with a rigorous mathematics.

To calculate hydrodynamic flows, we use an explicit form for the local equilibrium statistical operator. But, in contrast to Morozov's work which operates with the statistical operator of superfluid helium at the laboratory reference system, we construct it in the reference system, in which the condensate is motionless. This gives the essential simplification, because the superfluid component is stopped in the local frame of reference moving with $\mathbf{v}_s$, and the total current is carried by the normal component.

We conditionally separate the derivation of the two-fluid hydrodynamics into two stages. On the first one, by using the Heisenberg equation of motion for both normal and anomalous correlation functions, we derive the conservation relations for the densities of particles ρ, momentum $\mathbf{j}$, and energy ε, as well as the equation of motion for the superfluid velocity $\mathbf{v}_s$. On the second stage, we express hydrodynamic flows in the conservation relations in terms of the already introduced variables (ρ, $\mathbf{j}$, and ε).

2. Conservation Relations for Hydrodynamic Quantities

2.1. Equation of motion for correlation functions in the Wigner representation

Helium-4 is a typical Bose system with pair interaction. Its Hamiltonian in the second quantization representation has the following form (we set $\hbar = 1$ throughout this paper):

$$H = \int d\mathbf{r}\psi^+(\mathbf{r})\left(-\frac{1}{2m}\Delta\right)\psi(\mathbf{r})+ \tag{1}$$

$$+\frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\Phi(\mathbf{r}-\mathbf{r}')\psi^+(\mathbf{r})\psi^+(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}).$$

Here, $\psi^+(\mathbf{r})$ and $\psi(\mathbf{r})$ are the creation and annihilation operators, respectively, and $\Phi(\mathbf{r}-\mathbf{r}') = \Phi(|\mathbf{r}-\mathbf{r}'|)$ is the interaction potential.

To construct the hydrodynamics of systems with spontaneously broken symmetry, we should proceed from the extended system of correlation functions [6] which includes both normal and anomalous correlation functions. Therefore, we will start with a system of correlation functions in the form

$$\langle\psi^+(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle, \quad \langle\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle. \tag{2}$$

Here, the angular brackets indicate the average over the locally equilibrium ensemble, and the dependence of the creation and annihilation operators on the time is given through the Heisenberg representation, for instance,

$$\psi(\mathbf{r},t) = e^{iHt}\psi(\mathbf{r})e^{-iHt}.$$

We note that the average in (2) is treated as a quasiaverage [5]. For the sake of simplicity, we will not take the "ν–term" that breaks the symmetry of Hamiltonian (1) into account.

Another note concerns the anomalous correlation function. In article [3], Bogolyubov operated with the wave function of the condensate $\langle\psi(\mathbf{r},t)\rangle$. In contrast to Bogolyubov, we work with a pair anomalous average. The presence of the nonvanishing pair anomalous average $\langle\psi\psi\rangle$ is ensured by the violation of the particle number selection rule. Using the pair anomalous average allows us to introduce a mixed Wigner representation by analogy with the microscopic approach for the derivation of the two-fluid hydrodynamics for superconductors [7].

Using the Heisenberg's equation of motion

$$i\frac{\partial\psi(\mathbf{r},t)}{\partial t} = [\psi(\mathbf{r},t),H]_- =$$

$$= -\frac{1}{2m}\Delta\psi(\mathbf{r},t) + \int d\mathbf{r}'\Phi(\mathbf{r}-\mathbf{r}')\psi^+(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r},t),$$

we obtain the equations of motion for correlation functions (2).

These equations are as follows:

$$i\frac{\partial}{\partial t}\langle\psi^+(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle = \frac{1}{2m}\left(\Delta_1-\Delta_2\right)\langle\psi^+(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle-$$

$$-\int d\mathbf{r}'\left\{\Phi(\mathbf{r}_1-\mathbf{r}')-\Phi(\mathbf{r}_2-\mathbf{r}')\right\}\times$$

$$\times\langle\psi^+(\mathbf{r}_1,t)\psi^+(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r}_2,t)\rangle, \tag{3}$$

$$i\frac{\partial}{\partial t}\langle\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle = -\frac{1}{2m}\left(\Delta_1+\Delta_2\right)\langle\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle+$$

$$+\int d\mathbf{r}'\left\{\Phi(\mathbf{r}_1-\mathbf{r}')+\Phi(\mathbf{r}_2-\mathbf{r}')\right\}\times$$

$$\times\langle\psi^+(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle+$$

$$+\Phi(\mathbf{r}_1-\mathbf{r}_2)\langle\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle. \tag{4}$$

The next step will be a separation of gauge-noninvariant multipliers (in fact, we will use a reference system, in which the condensate is motionless). Such a separation of the phase has the form

$$\psi(\mathbf{r},t) \to \tilde{\psi}(\mathbf{r},t) = \psi(\mathbf{r},t)e^{im\chi(\mathbf{r},t)}.$$

The separation of the phase transforms the correlation functions in the following way:

$$\langle\psi^+(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle = e^{im(\chi(\mathbf{r}_2,t)-\chi(\mathbf{r}_1,t))}G(\mathbf{r}_1,\mathbf{r}_2;t),$$

$$\langle\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle = e^{im(\chi(\mathbf{r}_2,t)+\chi(\mathbf{r}_1,t))}F(\mathbf{r}_1,\mathbf{r}_2;t),$$

$$\langle\psi^+(\mathbf{r}_1,t)\psi^+(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r}_2,t)\rangle = e^{im(\chi(\mathbf{r}_2,t)-\chi(\mathbf{r}_1,t))}\times$$

$$\times\mathcal{D}^{(1)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}';t),$$

$$\langle\psi^+(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r}_1,t)\psi(\mathbf{r}_2,t)\rangle = e^{im(\chi(\mathbf{r}_2,t)+\chi(\mathbf{r}_1,t))}\times$$

$$\times\mathcal{D}^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}';t).$$

The functions G, F, $\mathcal{D}^{(1)}$, and $\mathcal{D}^{(2)}$ in the statistical equilibrium state are spatially homogeneous. In the nonequilibrium states, they are changed less than the spatially inhomogeneous functions.

Then the equations of motion for G and F are as follows:

$$\left\{i\frac{\partial}{\partial t} + m\dot{\chi}(\mathbf{r}_1,t) - m\dot{\chi}(\mathbf{r}_2,t)\right\} G(\mathbf{r}_1,\mathbf{r}_2;t) =$$

$$= -\frac{1}{2m}\left[(\hat{\mathbf{p}}_1 - m\mathbf{v}_s(\mathbf{r}_1,t))^2 - (\hat{\mathbf{p}}_2 + m\mathbf{v}_s(\mathbf{r}_2,t))^2\right] G(\mathbf{r}_1,\mathbf{r}_2;t) -$$

$$-\int d\mathbf{r}'\left\{\Phi(\mathbf{r}_1-\mathbf{r}') - \Phi(\mathbf{r}_2-\mathbf{r}')\right\}\mathcal{D}^{(1)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}';t), \quad (5)$$

$$\left\{i\frac{\partial}{\partial t} - m\dot{\chi}(\mathbf{r}_1,t) - m\dot{\chi}(\mathbf{r}_2,t)\right\} F(\mathbf{r}_1,\mathbf{r}_2;t) =$$

$$= \frac{1}{2m}\left[(\hat{\mathbf{p}}_1 + m\mathbf{v}_s(\mathbf{r}_1,t))^2 + (\hat{\mathbf{p}}_2 + m\mathbf{v}_s(\mathbf{r}_2,t))^2\right] F(\mathbf{r}_1,\mathbf{r}_2;t) +$$

$$+\int d\mathbf{r}'\left\{\Phi(\mathbf{r}_1-\mathbf{r}') + \Phi(\mathbf{r}_2-\mathbf{r}')\right\}\mathcal{D}^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}';t) +$$

$$+\Phi(\mathbf{r}_1-\mathbf{r}_2)F(\mathbf{r}_1,\mathbf{r}_2;t), \quad (6)$$

where $\mathbf{v}_s = \nabla\chi$ is the superfluid velocity (velocity of the condensate). Note that χ is the phase of a one-particle condensate. Recently (see article [8] and references therein), the presence of a pair (and higher) condensate in superfluid helium-4 was proved theoretically. If we take the higher condensates into account, we must assume the velocities of all condensates to be the same. In the theory of superconductivity, the typical anomalous average is $\langle\psi_\downarrow\psi_\uparrow\rangle$, but the superfluid velocity is defined as a gradient of the phase χ of the ψ-operator [6, 7, 9].

The transition to the equations of hydrodynamics is performed with the use of the expansion of Eqs. (5) and (6) in space gradients. This expansion can be simply realized by using the so-called mixed Wigner representation [6]. For this purpose, we introduce new variables

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2), \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$

After the Fourier transformation with respect to the relative coordinate $\mathbf{r}$, we obtain

$$f(\mathbf{r}_1,\mathbf{r}_2,t) \to f(\mathbf{R},\mathbf{r},t) = \int\frac{d\mathbf{p}}{(2\pi)^3}f(\mathbf{R},\mathbf{p},t)e^{i\mathbf{pr}},$$

and

$$\mathbf{r}_1 \to \mathbf{R} - \frac{i}{2}\nabla_{\mathbf{p}}, \qquad \mathbf{r}_2 \to \mathbf{R} + \frac{i}{2}\nabla_{\mathbf{p}},$$

$$\hat{\mathbf{p}}_1 \to \mathbf{p} - \frac{i}{2}\nabla_{\mathbf{R}}, \qquad \hat{\mathbf{p}}_2 \to -\mathbf{p} - \frac{i}{2}\nabla_{\mathbf{R}}. \quad (7)$$

Any function of $\mathbf{R} + i/2\cdot\nabla_{\mathbf{p}}$ can be understood in terms of its power-series expansion

$$f(\mathbf{R} + \frac{i}{2}\nabla_{\mathbf{p}}) = f(\mathbf{R}) + \frac{i}{2}\frac{\partial f(\mathbf{R})}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}} - \cdots. \quad (8)$$

Using procedures (7) and (8), the equations for correlation functions can be written as

$$\frac{\partial G_{\mathbf{p}}(\mathbf{R},t)}{\partial t} = m\dot{v}_{si}\frac{\partial G_{\mathbf{p}}(\mathbf{R},t)}{\partial p_i} + \left(\frac{p_i}{m} + v_{si}\right)\frac{\partial G_{\mathbf{p}}(\mathbf{R},t)}{\partial R_i} -$$

$$-\frac{\partial}{\partial R_j}\left(\frac{(p_i+mv_{si})^2}{2m}\right)\frac{\partial G_{\mathbf{p}}(\mathbf{R},t)}{\partial p_j} +$$

$$+\frac{\partial}{\partial R_j}\left(\frac{1}{2}\int d\mathbf{r}'\frac{\partial\Phi(r')}{\partial r'_i}r'_j\frac{\partial\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t)}{\partial p_i}\right). \quad (9)$$

$$\left(i\frac{\partial}{\partial t} - 2m\dot{\chi}\right)F_{\mathbf{p}}(\mathbf{R},t) = \left(\frac{\mathbf{p}^2}{m} + m\mathbf{v}_s^2 + \tilde{\Phi}(\mathbf{p})\right)F_{\mathbf{p}}(\mathbf{R},t) -$$

$$+ip_i\frac{\partial v_{si}}{\partial R_j}\frac{\partial F_{\mathbf{p}}(\mathbf{R},t)}{\partial p_j} + 2\int d\mathbf{r}'\Phi(r')\mathcal{D}^{(2)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t) +$$

$$+iv_{si}\frac{\partial F_{\mathbf{p}}(\mathbf{R},t)}{\partial R_i} + 2\frac{\partial}{\partial R_j}\int d\mathbf{r}'\Phi(r')r'_j\mathcal{D}^{(2)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t). \quad (10)$$

Here

$$G_{\mathbf{p}}(\mathbf{R},t) = \int d\mathbf{r}\langle\psi^{+}(\mathbf{R}-\frac{\mathbf{r}}{2},t)\psi(\mathbf{R}+\frac{\mathbf{r}}{2},t)\rangle e^{i\mathbf{pr}}, \qquad (11)$$

$$F_{\mathbf{p}}(\mathbf{R},t) = \int d\mathbf{r}\langle\psi(\mathbf{R}-\frac{\mathbf{r}}{2},t)\psi(\mathbf{R}+\frac{\mathbf{r}}{2},t)\rangle e^{i\mathbf{pr}}, \qquad (12)$$

$$\times\psi(\mathbf{r}',t)\psi(\mathbf{R}+\frac{\mathbf{r}}{2},t)\rangle e^{i\mathbf{pr}}, \qquad (13)$$

$$\dot{v}_{si} = \frac{\partial v_{si}(\mathbf{R},t)}{\partial t}, \quad \dot{\chi} = \frac{\partial\chi(\mathbf{R},t)}{\partial t},$$

and $\tilde{\Phi}(\mathbf{p})$ is the Fourier transform of the interaction potential.

In Eqs. (9) and (10), the second-order terms with respect to the space gradient (the terms proportional to $\nabla^2_{\mathbf{R}}$) were neglected, which corresponds to the approximation of the ideal hydrodynamics.

We call Eq. (9) the forming equation, because its use gives the conservation laws for hydrodynamic quantities. Moreover, in terms of (9), we will obtain the equation of motion for the superfluid velocity.

Let us pass to obtaining the differential conservation laws (balance equations).

2.2. Equation of motion for the superfluid velocity

Let us consider the zero order of the equation for the anomalous correlation function (10). This equation reads

$$\left(m\dot{\chi}(\mathbf{R},t) + \frac{m\mathbf{v}_s^2(\mathbf{R},t)}{2}\right)F_{\mathbf{p}}(\mathbf{R},t) =$$

$$= -\left(\frac{\mathbf{p}^2}{2m} + \frac{\tilde{\Phi}(\mathbf{p})}{2}\right)F_{\mathbf{p}}(\mathbf{R},t) - \int d\mathbf{r}'\Phi(r')\mathcal{D}^{(2)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t).$$

The separation of variables gives

$$m\dot{\chi}(\mathbf{R},t) + \frac{m\mathbf{v}_s^2(\mathbf{R},t)}{2} =$$

$$= -\frac{\left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}\tilde{\Phi}(\mathbf{p})\right)F_{\mathbf{p}}(\mathbf{R},t) + \int d\mathbf{r}'\Phi(r')\mathcal{D}^{(2)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t)}{F_{\mathbf{p}}(\mathbf{R},t)} \equiv$$

$$\equiv -\mu(\mathbf{R},t).$$

Hence,

$$m\dot{\chi}(\mathbf{R},t) + \frac{1}{2}m\mathbf{v}_s^2(\mathbf{R},t) + \mu(\mathbf{R},t) = 0 \qquad (14)$$

and

$$\left(\frac{\mathbf{p}^2}{2m} - \mu + \frac{\tilde{\Phi}(\mathbf{p})}{2}\right)F_{\mathbf{p}}(\mathbf{R},t) - \int d\mathbf{r}'\Phi(r')\mathcal{D}^{(2)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t) = 0.$$

Here, μ is the order of separation of the variables that coincide with the chemical potential.

Applying the operation $\nabla_{\mathbf{R}}$ to Eq. (14), we obtain the equation of motion for the superfluid velocity:

$$m\frac{\partial\mathbf{v}_s}{\partial t} + \nabla_{\mathbf{R}}\left(\frac{m{\mathbf{v}_s}^2}{2} + \mu\right) = 0. \qquad (15)$$

This first hydrodynamic equation shows that the superfluid is accelerated freely under applied fields. The remaining hydrodynamic equations are provided by the conservation relations for the particle density $\rho(\mathbf{R},t)$, momentum density $\mathbf{j}(\mathbf{R},t)$, and energy density $\mathcal{E}(\mathbf{R},t)$. These equations follow simply from the calculation by moments of the forming equation (9).

2.3. Equation for the particle density

By definition,

$$\rho(\mathbf{R},t) = m\langle\psi^{+}(\mathbf{R},t)\psi(\mathbf{R},t)\rangle = m\int\frac{d\mathbf{p}}{(2\pi)^3}G_{\mathbf{p}}(\mathbf{R},t).$$

After integrating (9) over $\mathbf{p}$, we find

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\mathbf{j} = 0, \qquad (16)$$

where

$$\mathbf{j}(\mathbf{R},t) = \int\frac{d\mathbf{p}}{(2\pi)^3}\mathbf{p}G_{\mathbf{p}}(\mathbf{R},t) + \rho\mathbf{v}_s \equiv \mathbf{j}_0 + \rho\mathbf{v}_s. \qquad (17)$$

Equation (16) is the equation of continuity for the particle density. The quantity $\mathbf{j}(\mathbf{R},t)$ is the momentum density, respectively, and $\mathbf{j}_0$ is the momentum density in a reference system, where the condensate is motionless. The calculation of $\mathbf{j}_0$ in explicit form will be performed in Section 3.

2.4. Equation for the momentum density

Using definition (17), we find that

$$\frac{\partial j_k}{\partial t} = \frac{\partial}{\partial t}(j_{0k} + \rho v_{sk}) =$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3}(p_k + mv_{sk})\frac{\partial G_{\mathbf{p}}(\mathbf{R},t)}{\partial t} + \rho\frac{\partial v_{sk}}{\partial t}.$$

Taking the moment of the forming equation (9) with respect to $\mathbf{p} + m\mathbf{v}_s$ and using the equation of motion for the superfluid velocity (15), we obtain

$$\frac{\partial j_k}{\partial t} + \frac{\partial \Pi_{kj}}{\partial R_j} = 0. \qquad (18)$$

The momentum density flow (stress tensor) is given by

$$\Pi_{kj} = \frac{1}{m}\int \frac{d\mathbf{p}}{(2\pi)^3}(p_k + mv_{sk})(p_j + mv_{sj})G_{\mathbf{p}}(\mathbf{R},t) -$$

$$-\frac{1}{2}\int d\mathbf{r}'\frac{\partial\Phi(r')}{\partial r'_k}r'_j\int\frac{d\mathbf{p}}{(2\pi)^3}\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t) =$$

$$= v_{sk}j_{0j} + v_{sj}j_{0k} + \rho v_{sk}v_{sj} + \Pi_{0kj}, \qquad (19)$$

where

$$\Pi_{0kj} = \frac{1}{m}\int\frac{d\mathbf{p}}{(2\pi)^3}p_k p_j G_{\mathbf{p}}(\mathbf{R},t) -$$

$$-\frac{1}{2}\int d\mathbf{r}'\frac{\partial\Phi(r')}{\partial r'_k}r'_j\int\frac{d\mathbf{p}}{(2\pi)^3}\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t). \qquad (20)$$

2.5. Equation for the energy density

By definition, the energy density of particles in the laboratory reference system is as follows:

$$\mathcal{E}(\mathbf{r},t) = \frac{1}{2m}\langle\nabla\psi^+(\mathbf{r},t)\nabla\psi(\mathbf{r},t)\rangle +$$

$$+\frac{1}{2}\int d\mathbf{r}'\Phi(|\mathbf{r}-\mathbf{r}'|)\langle\psi^+(\mathbf{r},t)\psi^+(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r},t)\rangle.$$

In the system of reference, where the condensate is motionless, the energy density is

$$\mathcal{E}(\mathbf{R},t) = \frac{1}{2m}\int\frac{d\mathbf{p}}{(2\pi)^3}(\mathbf{p}+m\mathbf{v}_s)^2 G_{\mathbf{p}}(\mathbf{R},t) +$$

$$+\frac{1}{2}\int d\mathbf{r}'\Phi(|\mathbf{R}-\mathbf{r}'|)\int\frac{d\mathbf{p}}{(2\pi)^3}\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t)$$

or

$$\mathcal{E} = \mathcal{E}_0 + \mathbf{j}_0\mathbf{v}_s + \frac{1}{2}\rho v_s^2, \qquad (21)$$

where

$$\mathcal{E}_0 = \frac{1}{2m}\int\frac{d\mathbf{p}}{(2\pi)^3}p^2 G_{\mathbf{p}}(\mathbf{R},t) +$$

$$+\frac{1}{2}\int d\mathbf{r}'\Phi(|\mathbf{R}-\mathbf{r}'|)\int\frac{d\mathbf{p}}{(2\pi)^3}\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t). \qquad (22)$$

By analogy to the previous subsections, we find

$$\frac{\partial\mathcal{E}}{\partial t} + \mathrm{div}\mathbf{Q} = 0. \qquad (23)$$

The energy flow is given by

$$\mathbf{Q} = \left(\mathcal{E}_0 + \mathbf{j}_0\mathbf{v}_s + \frac{1}{2}\rho v_s^2\right)\mathbf{v}_s + \frac{1}{2}v_s^2\mathbf{j}_0 + \mathbf{\Pi}_0\mathbf{v}_s + \mathbf{Q}_0, \qquad (24)$$

where

$$Q_{0k} = \frac{1}{2m}\int\frac{d\mathbf{p}}{(2\pi)^3}p^2 p_k G_{\mathbf{p}}(\mathbf{R},t) +$$

$$+\frac{1}{2m}\int d\mathbf{r}'\Phi(r')\int\frac{d\mathbf{p}}{(2\pi)^3}p_k\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t) - \qquad (25)$$

$$-\frac{1}{2m}\int d\mathbf{r}'\frac{\partial\Phi(r')}{\partial r'_j}r'_k\int\frac{d\mathbf{p}}{(2\pi)^3}p_j\mathcal{D}^{(1)}_{\mathbf{p}}(\mathbf{R},\mathbf{r}';t).$$

The system of equations (15), (16), (18), and (23) is a complete system of balance equations for superfluid helium-4.

3. Calculation of Hydrodynamic Flows

In the previous section, we have obtained a system of balance equations. These equations are nonclosed, because flows (17), (20), and (25) are unknown. When we have an explicit expression for the G-function, then the determination of the hydrodynamic flows is realized by calculation of the momentum integrals. In the case of superfluid helium, finding the G-function is impossible. Therefore, we must develop some "indirect" method to find flows (17), (20), and (25).

In finding the hydrodynamic flows, we used an explicit expression for the local equilibrium statistical operator. In contrast to work [4], we constructed the statistical operator in the reference system, where the condensate is motionless, which leads to some simplification.

The local equilibrium statistical operator that describes superfluid helium in the reference system, where the condensate is motionless, is as follows:

$$\hat{\varrho} = \exp\left\{\int d\mathbf{r}\beta(\mathbf{r})\left[\Omega(\mathbf{r}) - \hat{H}_0(\mathbf{r}) - \mathbf{u}\hat{\mathbf{P}}_0(\mathbf{r}) - \frac{\mu}{m}\hat{\rho}(\mathbf{r})\right]\right\}. \qquad (26)$$

In the local frame of reference moving with $\mathbf{v}_s$, the superfluid component is stopped, and the total current is carried by the normal component.

Therefore,

$$\mathbf{j}_0 = \langle \hat{\mathbf{P}}_0 \rangle = -\frac{\partial \Omega}{\partial \mathbf{u}} \equiv \rho_n \mathbf{u}. \tag{27}$$

Here, ρ_n is the normal fluid density.

Substituting (27) in (17), we find the momentum density (mass flow)

$$\mathbf{j} = \mathbf{j}_0 + \rho \mathbf{v}_s = \rho_n(\mathbf{v}_n - \mathbf{v}_s) + \rho \mathbf{v}_s =$$

$$= \rho_n \mathbf{v}_n + (\rho - \rho_n)\mathbf{v}_s \equiv \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s, \tag{28}$$

where $\rho_s = \rho - \rho_n$ is the superfluid density.

To find the stress tensor, we use a very elegant “scale transformation” method introduced by Bogolyubov [3].

Let us consider the transformation

$$r_i \to \lambda r_i, \quad r_k \to r_k \ (k \neq i).$$

The mean energy is thus transformed by the rule

$$\langle \hat{H}_0 \rangle \to \langle \hat{H}_{0\ \lambda} \rangle = \frac{1}{2m} \int \frac{d\mathbf{p}}{(2\pi)^3} \mathbf{p}_\lambda^2 G_{\mathbf{p}}(\mathbf{R}, t) +$$

$$+ \frac{1}{2} \int d\mathbf{r}' \Phi_\lambda(\mathbf{R} - \mathbf{r}') \int \frac{d\mathbf{p}}{(2\pi)^3} \mathcal{D}_{\mathbf{p}}^{(1)}(\mathbf{R}, \mathbf{r}'; t),$$

where

$$\mathbf{p}_\lambda^2 = \frac{1}{\lambda^2} p_i^2 + \sum_{k \neq i} p_k^2, \quad \Phi_\lambda(\mathbf{r}) = \Phi(\lambda r_i, r_k).$$

It is easy to show that

$$\frac{\partial \langle \hat{H}_{0\ \lambda} \rangle}{\partial \lambda}\Big|_{\lambda=1} = -\Pi_{0ii}.$$

Using the relationship

$$\frac{\partial \langle \hat{H}_{0\ \lambda} \rangle}{\partial \lambda} = \frac{\partial \Omega_\lambda}{\partial \lambda}, \text{ where } \Omega_\lambda = \Omega\left(\beta, \frac{\rho}{\lambda}, \lambda u_i, u_k\right),$$

we find

$$\Pi_{0ii} = -\frac{\partial \Omega_\lambda}{\partial \lambda}\Big|_{\lambda=1} = -\left(-\rho \frac{\partial \Omega}{\partial \rho} + u_i \frac{\partial \Omega}{\partial u_i}\right) = P + \rho_n u_i^2.$$

Here, $P = \rho \dfrac{\partial \Omega}{\partial \rho}$ is the pressure.

Since the selected direction in the system of reference, where the condensate is motionless, is $\mathbf{u}$, we have

$$\Pi_{0ik} = \rho_n u_i u_k + \delta_{ik} P. \tag{29}$$

Substituting (29) in (19), we find the final form of the stress tensor (momentum flow):

$$\Pi_{ik} = \Pi_{0ik} + v_{si} j_{0k} + v_{sk} j_{0i} + \rho v_{si} v_{sk} =$$

$$= \rho_s v_{si} v_{sk} + \rho_n v_{ni} v_{nk} + \delta_{ik} P. \tag{30}$$

To calculate the energy flow, we employ the obvious identity [4]

$$\langle [\hat{H}_0, \hat{S}]_- \rangle = 0, \tag{31}$$

where $\hat{S}$ is the entropy operator that is defined by the relation $\hat{\varrho} = \exp\{-\hat{S}\}$.

Using the explicit form of the local equilibrium statistical operator (26), identity (31) gives

$$\int d\mathbf{r} \beta(\mathbf{r}) \Big\langle [\hat{H}_0, \hat{H}_0(\mathbf{r})]_- - u_i [\hat{H}_0, \hat{P}_{0i}(\mathbf{r})]_- -$$

$$- \frac{\mu}{m} [\hat{H}_0, \hat{\rho}(\mathbf{r})]_- \Big\rangle = 0.$$

Inserting the Heisenberg equations of motion for $\hat{H}_0(\mathbf{r}), \hat{P}_{0i}$, and $\hat{\rho}(\mathbf{r})$, we obtain

$$\int d\mathbf{r} \beta(\mathbf{r}) \left\{ \nabla_i Q_{0\ i} - u_k \nabla_i \Pi_{0\ ik} - \frac{\mu}{m} \nabla_i j_{0\ i} \right\} = 0.$$

Due to the arbitrariness of the integration domain, we have

$$\nabla_i Q_{0\ i} = u_k \nabla_i \Pi_{0\ ik} + \frac{\mu}{m} \nabla_i j_{0\ i} + \frac{1}{\beta} \nabla_i g_i.$$

Substituting (27) and (29), we find

$$\mathbf{g} = S\mathbf{u},$$

$$\mathbf{Q}_0 = \left(\rho_n u^2 + \frac{\rho\mu}{m} + TS\right) \mathbf{u}. \tag{32}$$

Finally, the expressions for hydrodynamic flows have the form

$$\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \quad \mathbf{v}_n \equiv \mathbf{u} + \mathbf{v}_s,$$

$$\Pi_{ik} = \rho_n v_{ni} v_{nk} + \rho_s v_{si} v_{sk} + \delta_{ik} P,$$

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$$\mathbf{Q} = \left(\frac{v_s^2}{2} + \frac{\mu}{m}\right)\mathbf{j} + TS\mathbf{v}_n + \rho_n \mathbf{v}_n(\mathbf{v}_n \cdot (\mathbf{v}_n - \mathbf{v}_s)).$$

These hydrodynamic flows coincide with those in the two-fluid hydrodynamics of Landau [1].

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МОДИФІКОВАНЕ БОГОЛЮБОВСЬКЕ ВИВЕДЕННЯ ДВОРІДИННОЇ ГІДРОДИНАМІКИ

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Р е з ю м е

У роботі представлено послідовний мікроскопічний вивід дворідинної гідродинаміки надплинного гелію-4 в ідеальному наближенні. Відправною точкою у нашому формалізмі є система гайзенбергівських рівнянь руху для нормальної та аномальної кореляційних функцій. Використання мішаного представлення Вігнера дозволяє легко і з математичною строгістю виконати розклад рівнянь руху для кореляційних функцій за градієнтами. Для обчислення гідродинамічних потоків у роботі побудовано локально-рівноважний статистичний оператор надплинної рідини в системі відліку, пов'язаній з конденсатом.