TO QUESTION OF CREATION THE WHITE DWARFS THEORY

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It is of interest to inquire what the relation is between Stoner's minimum energy method and Chandrasekhar's equation of gravitational equation. Treating Stoner's minimum energy principle as a variational problem in which the total energy is a functional of the density, and this density is a variable function of the radial density, this variational approach leads to the quantum mechanical ground state of an electron gas in the gravitational field of the ions, which maintain charge neutrality. This connection explains why Stoner and Chandrasekhar obtained the same relations for the density and mass of the star as functions of fundamental constants, but with somewhat different dimensionless quantities. In particular, I will show that the solution to the generalized form of Stoner's variational problem for the minimum of the total energy of a dense star leads to the differential equation of gravitational equilibrium which Chandrasekhar applied in his work. I have not found any evidence, however, that either Stoner or Chandrasekhar were aware of this connection.

The total energy *E* of a zero temperature dense star supported entirely by degeneracy pressure against the gravitational attractive forces can be written as a functional of the mass density distribution r integrated over the volume of the star,

$$E = \int dv[\varepsilon(\rho) - u(p, r)]\,,$$

where $\varepsilon(\rho)$ is the internal energy given as a function of of the mass density r by Stoner' relativistic equation of state for a electron degenerate gas, $u(p, r)$ is gravitational energy

$$u(p, r) = -\frac{1}{2} G \int dv' \frac{\rho(r')\rho(r)}{|r - r'|}\,,$$

and *G* is Newton's gravity constant.

The equilibrium distribution r as a function of position *r* can be determined by evaluating the minimum of *E* , subject to the condition that the total mass *M* = ò*dv*r is fixed. Assuming that r depends only on the radial distance *r* from the center of the star, this variational problem leads to the differential equation for gravitational equilibrium,

$$\frac{dP}{dr} = -G \frac{M(r)\rho(r)}{r^2}$$

It is well known that the theory of white dwarfs (the Chandrasekhar limit) was created by R. Fowler's graduate student S. Chandrasekhar. In 1926, R. Fowler drew attention to the fact that it is appropriate to use Fermi-Dirac statistics in the astrophysics of stars. This idea was implemented by S. Chandrasekhar in his polytropic theory of white dwarfs.

Calculated values for the limit vary depending on the nuclear composition of the mass. Chandrasekhar gives the following expression, based on the equation of state for an ideal Fermi gas:

$$M_{limit} = \frac{\omega_3^0}{2}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{(\mu_e m_H)^2}, \quad (1)$$

where: $\hbar$ is the Dirac or reduced Planck constant; c is the speed of light; μ_e is the average molecular weight per electron, which depends upon the chemical composition of the star; m_H is the mass of the hydrogen atom; $\omega_3^0 \approx 2.018236$ is a constant connected with the solution to the Lane–Emden equation

As $\left(\frac{\hbar c}{G}\right)^{1/2}$ is the Planck mass m_P, the limit is of the order of

$$\frac{m_P^3}{m_H^2}. \quad (2)$$

The limiting mass can be obtained formally from the Chandrasekhar's white dwarf equation by taking the limit of large central density.

However, in 1932 L. Landau declared that he was using the ideas of work in the astrophysics of stars. Practically, paper is an short review of the works of Fowler and Chandrasekhar], although some believe that Landau actually came to this himself. However, it should be noted that he gives the same limit for white dwarfs as Chandrasekhar without any conclusions or references.

INTRODUCTION

The existence of a mass limit for white dwarfs is usually attributed solely to the late astrophysicist Subrahmanyan Chandrasekhar, and this limit is named after him. But as is often the case, the history of this discovery is more nuanced. In this note I will show that the existence of a maximum mass was first established by Edmund C. Stoner, a physicist who began experimental research under the supervision of Rutherford at the Cavendish in Cambridge, but later switched to theoretical work. Rutherford recommended Stoner to a position at the Physics department of the University of Leeds where he spent his entire career.

According to G. Cantor, he was "probably the leading Cavendish-trained theoretical physicist of the 1920's ", although he learned theory mostly on his own, and became known for his work on magnetism. Unfortunately, Stoner suffered from diabetes and poor health which restricted his travels, and this may account for the fact that he did not receive wider recognition for his achievements.

In 1924 Stoner wrote a paper on the distribution of electrons among atomic levels[5]. In the preface of the fourth edition of his classic book, "Atomic Structure and Spectral Lines", Arnold Sommerfeld gave special mention to " einen grossen Fortschritt [a great advancement]" brought about by Stoner's analysis, which then came to the attention of Wolfgang Pauli, and played and important role in his formulation of the exclusion principle in quantum physics Therefore, it is not surprising that Stoner's interest in white dwarfs was aroused by Ralph H. Fowler's suggestion that the exclusion principle could be applied to solve a major puzzle, the origin of the extreme high density of white dwarfs, which could not be explained by classical physics. Eddington expressed this puzzle as follows:

`` I do not see how a star which has once got into this compressed state is ever going to go out of it... The star will need energy in order to cool…It would seem that the star will be in an awkward predicament when its supply of subatomic energy fails. Imagine a body continually losing heat but with insufficient energy to grow cold ! " .

At the time, the conventional wisdom was that the source of internal pressure which maintained all stars in equilibrium against gravitational collapse was the internal pressure of the matter composing the star which had been heated into a gas presumably, according to Eddington, by “subatomic energy’’. But when this supply of energy is exhausted and the star cools, Fowler proposed that a new equilibrium would ensue, even at zero temperature, due to the “degeneracy” pressure of the electrons caused by the exclusion principle. Fowler, however, did not attempt to determine the equilibrium properties of such a star which he regarded as “strictly analogous to one giant molecule in the ground state''. Apparently he was unaware that at the time, Llewellyn H. Thomas had developed a mathematical method to solve this problem in atomic physics[12]. Subsequently, Stoner developed a novel minimum energy principle to obtain the equilibrium properties of such dense stars[13], and by applying Fowler's non-relativistic equation of state for a degenerate electron gas in a constant density approximation, he found that the density increases with the square of the mass of the star[14]. In such a gas the mean momentum of an electron is proportional to the cube root of the density (see Appendix I), and Wilhem Anderson, a privatdozent at Tartu University, Estonia, who had read Stoner’s paper, noticed that for the mass of a white dwarf comparable to or higher than the mass of the Sun, the density calculated from Stoner’s non-relativistic mass-density relation implied that the electrons become relativistic. Hence, Anderson concluded that in this regime, this relationgave “gröblich falschen Resultaten [gross false results]” for the properties of a white dwarf. He attemped to extend the equation of state of a degenerate electron gas to the relativistic domain, but he gave an incorrect formulation which, fortuitously, indicated that Stoner’s minimum energy principle implied a maximum value for the white dwarf mass. Alerted by Anderson’s paper, Stoner then derived the correct relativistic equation of state[16], and re-calculated, in a constant density approximation, the properties of white dwarfs for arbitrary densities[17]. Thus, he obtained, now on solid theoretical grounds, the surprising result that when the density approaches infinity, the mass of the star reaches a maximum value.

Two years after the appearance of the first paper[13] by Stoner on the " limiting density of white dwarfs", Chandrasekhar published a paper[18] with a similar title "arriving at the order of magnitude of the density of white stars from different considerations". This paper was communicated by Fowler to the Philosophical Magazine . Since the non-relativistic pressure - density relation for a degenerate electron gas is a power law with exponent 5/3 (see Appendix I), Chandrasekhar realized - from having read Eddington's book "The Internal Constitution of the Stars"[11] , which he had obtained as an essay prize - that the solution of the differential equation for gravitational equilibrium of a low mass white dwarf was the Lande-Emde polytrope with index n=3/2. This solution leads to the same mass – density relation previously found by Stoner in the uniform density approximation, but with a proportionality coefficient smaller by a factor about two . Meanwhile, Stoner, in collaboration with Frank Tyler, also calculated the minimum energy of a white dwarf assuming a density distribution corresponding to the n=3/2 polytrope [19] obtaining the same result as Chandrasekhar , and somewhat earlier Edward A. Milne also had carried out this calculation [20]. In his paper Chandrasekhar ignored "relativistic-mass corrections", because he did not yet know how to incorporate them, while Stoner had shown

that for the white dwarf companion of Sirius these corrections gave a density almost an order of magnitude larger than the non-relativistic calculation. In his recollections[21], however, Chandrasekhar remarks that he had found that the degenerate electrons become relativistic [22] for white dwarfs with masses which are comparable or larger than the mass of the Sun. His calculation in the extreme relativistic limit appeared separately in a very short paper (two pages long) on " the maximum mass of ideal white dwarfs [23]. Again, Chandrasekhar was able to obtain his result with great ease, because the relevant solution of the differential equation for gravitational equilibrium for the extreme relativistic equation of state of a degenerate electron, which has an exponent 4/3 (see Appendix I),

corresponds to the the *n* = 3 Lane-Emde polytrope solution, which also appearsin Eddington's book . It turns out that for *n* = 3 the mass is independent of thecentral or mean density of the star. Chandrasekhar acknowledged that his result was in surprising "agreement'' with Stoner's result , but he also claimed, without giving any proof, that the critical mass was a maximum. Later, in an interview with Spencer Weart [24], Chandrasekhar acknowledged that

"…at first I didn't understand what this limit meant and I didn't know how it would end , and how it related to the 3/2 low mass polytropes. But all that I did when I was in England and wrote my second paper on it".

But a proof that the critical mass is a maximum already had been given in the uniform density approximation by Stoner, who also had shown analytically that the mass of a white dwarf is a monotonically increasing function of the density which is finite at infinite density, while it took Chandrasekhar several additional months before he found a rough argument to show that at the critical mass the density becomes infinite. But the fact that he was aware of Stoner’s analysis was left unmentioned, although it is clear that it must have given him confidence in the validity of his result.

Stoner's fully relativistic analytic solution, in the uniform density approximation (see Appendix I), for the mass-radius dependence of the dense stars is shown graphically in Fig. 1. His result is compared with ten numerical calculations, shown by circles, which Chandrasekhar obtained five years later by integrating numerically the differential equations of gravitational equilibrium with Stoner's relativistic pressure-density equation of state.

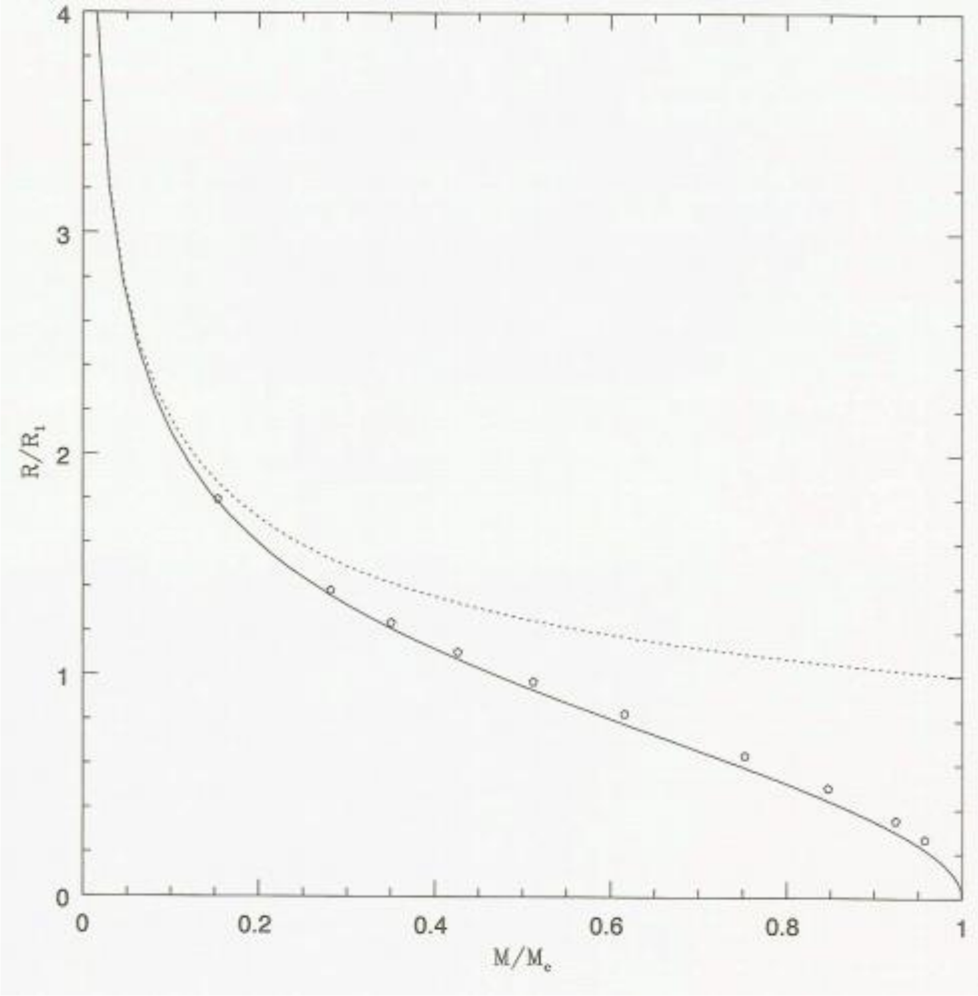

Fig. 1. The dark line is a plot of the scaled radius, $R/R1$ vs. scaled mass, M/Mc of Stoner's 1930 analytic solution in the uniform density approximation. The circles are the solutions published in 1935 by Chandrasekhar, who numerically integrated the equations of gravitational equilibrium using Stoner's pressure-density relativistic equation of state. The mass is given in units of the critical mass Mc , and the radius in units of a length R_1 for which $(M/Mc)(R/R\ )^3 \sim 1$ in the non-relativistic limit, $(M/Mc) << 1$. The dashed line is the non-relativistic solution $R/R1 = (MC/M\ )^{1/3}$.

This remarkable agreement is surprising, because Stoner's result was based on the uniform density approximation, while Chandrasekhar’s was obtained by integrating the equations of gravitational equilibrium. The main difference is in the scales of mass and of length, e.g. Chandrasekhar's critical mass M_c is 20 % smaller that Stoner's. Before 1935, following ideas of Milne, Chandrasekhar had developed only a crude composite model for a white dwarf [26] in which the non-relativistic approximation was assume to be valid for increasing mass until the central pressure became equal to the pressure given by the extreme relativistic equation at the same density. For a larger mass, he applied this relativistic equation to a central region of the star, and the non-relativistic equation for an external region of the star bounded by a surface defined when these two equations gave the same pressure at equal densities.

Stoner was encouraged by Arthur S. Eddington, the foremost astrophysicist at that time, to pursue the implication of his relativistic equation of state on the maximum density and temperature of white dwarfs as a function of density, and he communicated Stoner's two papers on this subject to the Monthly Notices of the Royal Astronomical Society[28][29] [30]. Eddington's 1932 correspondence with Stoner (see Appendix II and Fig. 2) deepens further the mystery why several years later, in a well known public attack on Chandrasekhar's similar work on white dwarfs [31], Eddington unexpectedly rejected the relativistic equation of state, and the profound implications of the existence of a white dwarf mass limit[32] [33] for the fate of stars with masses exceeding this limit[34]. Apparently Eddington had found that relativistic degeneracy was incompatible with his fundamental theory, and later confessed to Chandrasekhar that he would have to abandon this theory if relativitivistic degeneracy was valid[35] .

Eddington's criticisms were entirely unfounde but his enormous prestige led to the acceptance of his views by many in the astronomical community, and to an early rejection of Chandrasekhar's work. After Eddington questioned the validity of the relativistic equation of state for a degenerate electron gas, Chandrasekhar went for support to several of the great pioneers of the modern quantum theory, including Dirac who was in Cambridge, and to Bohr and Rosenfeld who he had met during a visit at Bohr's Institute in Copenhagen. They assured him of the validity of the relativistic equation of state, and advised him to ignore Eddington's objections, but Chandrasekhar continued relentlessly to pursue this matter, writing a paper with Christian Meller on relativistic degeneracy[40], and persuading Rudolf Peierls to give another proof of its validity. During this controversy, however, Chandrasekhar apparently did not mention Stoner and his earlier derivation of this equation, which is neither referenced in his paper with Meller nor in the paper by Peierls. In an Appendix to the first paper in which he applied Stoner's equation, he claimed to offer a "simpler derivation" of it, but it turned out to be essentially the same one given by Stoner. Here Chandrasekhar did give an acknowlegdment to Stoner with the remark that " this equation has been derived by Stoner (among others)", but the "others" remain unidentified, because they don't exist. He also mentioned " that Stoner had previously made some calculations concerning the (p, r) relations for a degenerate gas", neglecting to give reference to Stoner's paper[28] where a derivation of this pressure-density relation and his numerical tables appeared. For several more years Stoner continued to work on the equation of state for finite temperatures, publishing extensive tables of Fermi-Dirac functions which later turned out to be also very useful for improved calculations of the properties of white dwarfs . Chandrasekhar also did not mentioned that an independent derivation in 1931 of the critical mass of dense stars was given by Lev Landau, who apparently was unaware of Stoner's work. Landau, however could not have known of Chandrasekhar's work which appeared only after Landau had submitted his work for publication. Nevertheless, in his "historical notes", Chandrasekhar complained "the tendency in some current literature" to give Landau priority in this discovery, and never gave reference to Landau's work.

Later on, in his 1939 book [45] on stelllar book where he reproduced his work on white dwarfs, Chandrasekhar mentioned that the "equation for the internal energy of an electron gas" was derived by E. C. Stoner (p. 361), but again he neglected to refer to Stoner's explicitly derivation of the pressure-density relation, and his numerical tables for such a gas[28], although in 1934 he had to reproduce these tables with higher accuracy, because these tables were essential for his numerical integrations of the differential equations for gravitational equilibrium. He claimed (p. 422) that " the existence of this limiting mass was first isolated by Chandrasekhar , though its existence had been made apparent from earlier considerations by Anderson and Stoner …". One is left wondering, however, what he meant by this assertion. I have found two other occasions when he used the word "isolate'', which may give a clue to its meaning in the present context. In his book "Eddington , the most distinguished astrophysicst of this time " (Cambridge Univ. Press, Cambridge 1983), Chandrasekhar stated that when Eddington calculated the relation between mass and pressure in a star, he did not "isolate'' its dependence on natural constants, "a surprising omission in view of his later preoccupations with natural constants''. Likewise, in his Nobel speech [31], Chandrasekhar remarked that an inequality, given as Eq. (14), had "isolated'' the combination of natural constants of the dimension of mass. But in this sense, it was Stoner and not Chandrasekhar who first "isolated'' the limiting mass, because Stoner explicitly gave the dependence of this mass on natural constants. In his ``Biographical Notes'' (p. 451) where he gives a reference to only two of the five papers of Stoner on the properties of white dwarfs, Chandrasekhar's merely comments that in these papers "Stoner makes some further applications of Fowler's ideas'' , not giving the reader any idea of the important concepts and results regarding the properties of white dwarfs contained in these seminal papers. By such obfuscation, Chandrasekhar gave rise to the current neglect of Stoner's work.

In Kamesh Wali's excellent biography of Chandrasekhar, Stoner, is not mentioned even once, and his name also does not appear in Spencer Weart's transcript of his lengthy interview with Chandrasekhar in 1977. More recently, in his book "The Empire of Stars', Arthur Miller remarks that "it was indeed extraordinary that a nineteen-year-old Indian youth [Chandrasekhar] had managed to make a discovery that had eluded the great minds of European astrophysics'' (p.14) . Although Miller briefly refers to Anderson and to Stoner, he claimed that they "had never examined the ramifications'' of the relativistic equation of state (p. 133). But as we have shown here, with respect to Stoner Miller's claim is incorrect. In 1983 Chandrasekhar was awarded the Nobel prize, but in his acceptance speech, which mainly is a historical review of his work on white dwarfs, he did not include a single reference to Stoner. This general neglect of Stoner's seminal work on white dwarfs helps explain why, with a few notable exceptions, Stoner's contributions and his priority in the discovery of the maximum mass of white dwarfs have been forgotten now.

It is of interest to inquire what the relation is between Stoner's minimum energy method and Chandrasekhar's equation of gravitational equation. Treating Stoner's minimum energy principle as a variational problem in which the total energy is a functional of the density, and this density is a variable function of the radial density, this variational approach leads to the quantum mechanical ground state of an electron gas in the gravitational field of the ions, which maintain charge neutrality. This connection explains why Stoner and Chandrasekhar obtained the same relations for the density and mass of the star as functions of fundamental constants, but with somewhat different dimensionless quantities. In particular, I will show that the solution to the generalized form of Stoner's variational problem for the minimum of the total energy of a dense star leads to the differential equation of gravitational equilibrium which Chandrasekhar applied in his work. I have not found any evidence, however, that either Stoner or Chandrasekhar were aware of this connection.

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$$\frac{dP}{dr} = -G\frac{M(r)\rho(r)}{r^2}$$

here *P* = r*d*e/*d*r - e is the pressure, and $M(r) = 4\pi\int r^2\frac{M(r)\rho(r)}{r^2}dr$ is the mass inside the radius *r*. In the uniform density approximation, the solution of Stoner's minimum energy principle gives the relation

$P = (3/20\pi)GM^2/R^4$,

where *P* is the mean pressure, *M* is the mass and *R* is the radius of the star. Stoner's relativistic equation of state for the pressure - density relation of a degenerate electron gas was first given in the form $P = Ax^4F(x)$, where

$$F(x) = \frac{1}{8x^3}\left[\frac{3}{x}\log\left(x+\sqrt{1+x^2}\right)+\sqrt{1+x^2}\left(2x^3-3\right)\right],$$

and $x = \frac{3nh}{8\pi mc}$, Here *n* is the electrton density $n = \frac{3M}{4\pi R^3 m_H\mu}$, m_H is the proton mass, *h* is Planck's constant, *c* is the velocity of light , m is the molecular weight and $\frac{8\pi m^4c^5}{3h^3}$.

Hence Stoner's analytic solution for the mass M of a a white dwarf takes the form $M = M_c\left(4F(x)\right)^{3/2}$. In the limit of small density $x =$ 1, $F(x) = x/5$, and $P = (1/20)(3/\pi)^{2/3}(h^2/m)n^{5/3}$, which corresponds to Fowler's result for the pressure-density relation in the non-relativistic limit. In this limit we recover Stoner's original relation that the density n is proportional to the square of the mass M of the star, $n = (10\,\pi/3)(mc/h)^3(M/M_s)^2$. The maximum momentum of the electrons is $p = (mc)x$, and therefore when x is of order one or larger the effects of relativity become important, as was first pointed out by Anderson, and independently by Chandrasekhar. In the limit of infinite density, $x \to \infty$, $F(x) \to ¼$, which gives $P = (1/8)(3/\pi)^{1/3}n^{4/3}$, and $M = M_c$, with Stoner's critical mass expressed in terms of some of the fundamental constants of nature.

$$M_{cS} = \left({}^{3}/_{16\pi}\right)\left({}^{5hc}/_{2G}\right)^{3/2}(m_H\mu)^{-2}.$$

Chandrasekhar's result for the critical mass, expressed in terms of fundamental constants, corresponds

$$M_{cCh} = u\left({}^{\sqrt{6}}/_{8\pi}\right)\left({}^{hc}/_{G}\right)^{3/2}(m_H\mu)^{-2},$$

where $u = 2.018...$ is a constant obtained bynumerically integrating the equation of gravitational equilibrium for an $n = 3$ polytrope. It can be readily verified that the critical mass evaluated with a mass density distribution corresponding to an $n = 3$ polytrope is 20% smaller than for a uniform density distribution. In other words

$$\frac{M_{cS}}{M_{cCh}} = 1.2.$$

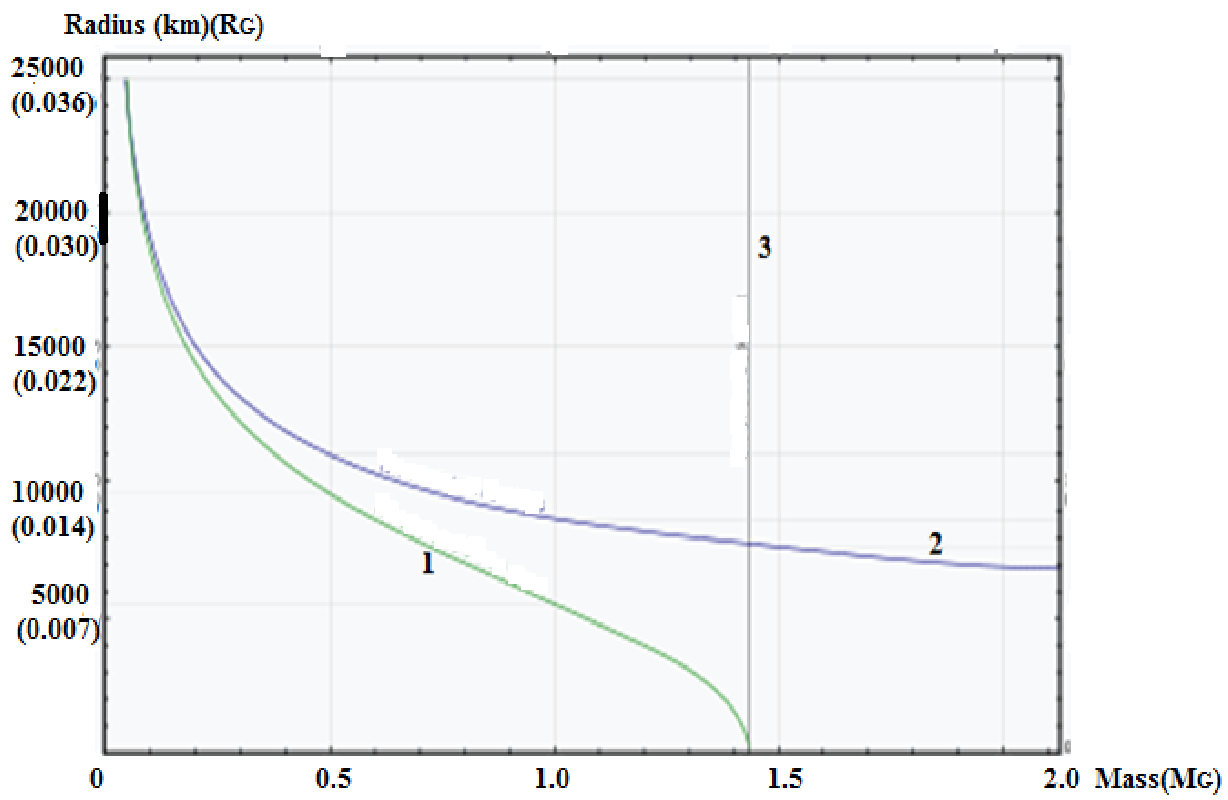

Fig. 2. Radius–mass relations for a model white dwarf. 1 – Using the general pressure law for an ideal Fermi gas; 2 – Non-relativistic ideal Fermi gas; 3 – Ultrarelativistic limit.

COMCLUSIONS

1. A. Eddington initiated the study of white dwarfs. He also introduced the name white dwarfs and demonstrated the expediency of using Lane and Emden formalisms. At that time, only two white dwarfs were known: 40 Eridanus B and B Sirius.
2. Stoner, R. Fowler, and S. Chandrasekhar developed the feasibility of using relativistic simulations to model the electron gas, including its degeneracy.
3. V. Pauli used Stoner's results in 2024 in formulating his principle for electrons.
4. Chandrasekhar developed a theory of the layered structure of a white dwarf (degenerate inside the star and non-degenerate in the outer layers).
5. Landau's 1932 work shows the level of understanding of this problem. In addition, considering the fact that Chandrasekhar asked R. Peierls to support him in the dispute with Eddington, it is more likely to be this support than the original research.

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Дякую за Увагу !

Thank You for Attention !