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# APPROXIMATION OF THE FUNCTIONS FROM SOBOLEV'S CLASSES IN UNIFORM AND INTEGRAL METRICS

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**1. Formulation of the problem.** Let $C$, $L_\infty$ and $L$ be the spaces of $2\pi$-periodic accordingly continuous, measurable, essentially limited and summable functions on the period with norms $\|f\|_C = \max\limits_t |f(t)|$; $\|f\|_\infty = \operatorname*{ess\,sup}\limits_t |f(t)|$; $\|f\|_L = \|f\|_1 = \int\limits_{-\pi}^{\pi} |f(t)|dt$.

Let $\Lambda = \{\lambda_\delta(k)\}$ be a plurality of the functions, which are dependent on $k = 0, 1, 2, \dots$ and on the parameter $\delta$, which is changed on the plurality $E_\Lambda \subseteq R$, which has at least one limit point. Let also $\lambda_\delta(0) = 1$, $\forall \delta \in E_\Lambda$. It is worthy to note, that in case, when $\delta = n,\ n \in N$, the numbers $\lambda_\delta(k) =: \lambda_{n,k}$ are the elements of numerical right-angled matrix $\Lambda = \{\lambda_{n,k}\}$ $(n, k = 0, 1, ...; \lambda_{n,0} = 1, n \in N \cup \{0\})$, and when besides that $\lambda_{n,k} \equiv 0, k > n$ — are the elements of numerical triangular matrix. With the help of the

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plurality $\Lambda$ every summable function $f(x)$, in view of its series of Fourier

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$$

we will put in accordance the series

$$\frac{a_0}{2}\lambda_\delta(0)+\sum_{k=1}^{\infty}\lambda_\delta(k)(a_k\cos kx+b_k\sin kx),\ \ \delta\in E_\Lambda.$$

If this series at every $\delta\in E_\Lambda$ is the series of Fourier of the summable function, then we will denote it by $U_\delta(f;x;\Lambda)$, and in case when $\delta=n, n\in N$ by $U_n(f;x;\Lambda)$. So, any plurality $\Lambda$ sets the method of the construction of operators $U_\delta(f;x;\Lambda)$. In this case it is also said, that plurality $\Lambda$ determines the concrete method ($\Lambda$-method) ) of summing up Fourier series.

Denote by $W_p^r$, $p=1,\infty$, the plurality of $2\pi$-periodic functions, which have absolutely continuous derivatives including to $(r-1)$-st order and $\|f^{(r)}\,(t)\,\|_p\leq 1$ (see, for example, [1]).

If the succession $\{\lambda_\delta(k)\}_{k=\overline{0,\infty}}$ is so that the series

$$\frac{1}{2}+\sum_{k=1}^{\infty}\lambda_\delta(k)\cos kt$$

is the series of Fourier of the summable function, then (analogically to [2, p. 46]) almost everywhere the equality will take place

$$U_\delta(f;x;\Lambda)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x+t)K_\delta(t;\Lambda)dt, \tag{1}$$

where

$$K_\delta(t;\Lambda)=\frac{1}{2}+\sum_{k=1}^{\infty}\lambda_\delta(k)\cos kt. \tag{2}$$

The sum about finding asymptotic equalities for the quantity

$$\mathcal{E}\left(\mathfrak{N};U_\delta(\Lambda)\right)_X=\sup_{f\in\mathfrak{N}}\|f\left(x\right)-U_\delta\left(f;x;\Lambda\right)\|_X\,,$$

where $X$ is normed space, $\mathfrak{N}\subseteq X$ — is a set class of functions, $U_\delta(f;x;\Lambda)$, $\delta\in E_\Lambda$, – are the operators, which are generated by concrete method $U_\delta(f,\Lambda)$ of summing up the series of Fourier, we will call the problem of Kolmogorov-Nikolsky, following A. I. Stepanec [3, p. 198].

If the function $\varphi(\delta)=\varphi(\mathfrak{N};\delta)$ was found in the explicit form, that at $\delta\to\delta_0$ (where $\delta_0$ is the limit point of the plurality $E_\Lambda$)

$$\mathcal{E}\left(\mathfrak{N};U_\delta(f,\Lambda)\right)_X=\varphi\left(\delta\right)+o\left(\varphi\left(\delta\right)\right),$$

we will say that, that the problem of Kolmogorov-Nikolsky is solved for class $\mathfrak{N}$ and operator $U_\delta(f,\Lambda)$ in the metric of space $X$.

The existence of the tight connection between quantities $\mathcal{E}(W_\infty^r;U_n)_C$ and $\mathcal{E}(W_1^r;U_n)_1$ was found out by S.M. Nikolsky [4]. Particularly, he showed, that the inequality takes place

$$\mathcal{E}(W_1^r;U_n)_1 \le \mathcal{E}(W_\infty^r;U_n)_C, \tag{3}$$

which for the series of important triangular methods of summing up Fourier series of becomes exact or asymptotic equality.

S.B. Stechkin and S.A. Telyakovskii [5] examined the arbitrary triangular methods of summing up the series of Fourier and showed that in most cases the quantities $\mathcal{E}(W_\infty^r;U_n)_C$ and $\mathcal{E}(W_1^r;U_n)_1$ are asymptotically equal.

V.P. Motornyj [6] set up inequality of the kind of (3) for the arbitrary triangular methods of the summing up the series of Fourier $U_n$ on the classes of functions $W^r_{\beta,1}H_\omega$, $W^r_{\beta,\infty}H_\omega$, and also he showed several cases, when those inequalities become exact or asymptotic equalities. Particularly he proved the equality

$$\mathcal{E}\left(W_1^r;U_n(\Lambda)\right)_1 = \mathcal{E}\left(W_\infty^r;U_n(\Lambda)\right)_C, \; r \ge 1, \tag{4}$$

if

$$K_n\left(t;\Lambda\right) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_{n,k}\cos kt \ge 0.$$

It is worthy to note, that all enumerated before the results, are connected with triangular $\Lambda$-methods of summing up the series of Fourier. As to the right-angled $\Lambda$-methods or $\Lambda$-methods, which are determined by plurality $\Lambda = \{\lambda_\delta(\cdot)\}$ of continuous on $[0;\infty)$, dependent on real parameter $\delta$ functions, the results more modest here.

In this case it is necessary to mention the results of the work by famous Polish mathematician P.Pych. In [7] she showed, that the statement takes place:

Theorem 1.1. *Suppose that*

$$Q_1\left(t,\delta\right) = \sum_{k=1}^{\infty} \frac{1-\lambda_\delta(k)}{k}\sin kt$$

*vanishes only at the points* $t = k\pi \;\; (k = 0, \pm1, \pm2, \ldots)$ *for any* $\delta \in E_\Lambda$. *Then, given an integer* $r \ge 1$,

$$\mathcal{E}(W_1^r;U_\delta)_1 = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^{k(r+1)}\,\frac{1-\lambda_\delta(2k+1)}{(2k+1)^{r+1}} = \mathcal{E}(W_\infty^r;U_\delta)_C.$$

But this theorem doesn't give the possibility to get the solving of the problem of Kolmogorov–Nikolsky in the integral metric for many linear methods, as for example, for the method of Poisson's biharmonic integral.

The aim of this work is to get analogical to (4) equalities for quantities $U_\delta(f;x;\Lambda)$, which are set with help of correlation (1).

## 2. Approximation of the functions from Sobolev's classes in uniform and integral metrics.

Theorem 2.1. *If the kernel* $K_\delta\left(t;\Lambda\right)$ *(see (2)) is summable and nonnegative, then, when* $r \in N$, *the equality takes place*

$$\mathcal{E}\left(W_1^r;U_\delta(\Lambda)\right)_1 = \mathcal{E}\left(W_\infty^r;U_\delta(\Lambda)\right)_C. \tag{5}$$

*Proof.* As for an arbitrary $r=1,2,\dots$ and $k\leq r+1$, according to [6, p. 20], the correlation is correct

$$\sup_{f\in W_1^r}\omega_k(f;t)_1=\sup_{f\in W_\infty^r}\omega_k(f;t)_C=\omega_k\left(\varphi_r;t\right),$$

where $\omega_k(f;t)$ is a module of continuity of order $k$ of the function $f(t)$, and $\varphi_r(t)$ is the $r$-th periodic integral of the function $\operatorname{sign}\sin t$ with the middle meaning on the period, which is equaled zero, then taking into consideration the equality

$$f(x)-U_\delta(f;x;\Lambda)=\frac{1}{\pi}\int\limits_0^\pi\left(f(x+t)-2f(x)+f(x-t)\right)K_\delta\left(t;\Lambda\right)dt,$$

we will have, that

$$\mathcal{E}\left(W_\infty^r;U_\delta(f;\Lambda)\right)_C\leq\frac{1}{\pi}\int\limits_0^\pi\sup_{f\in W_\infty^r}\omega_2(f;x)K_\delta(t;\Lambda)dt=\frac{1}{\pi}\int\limits_0^\pi\omega_2(\varphi_r;t)K_\delta(t;\Lambda)dt.$$

Then taking into consideration the equality (8) from [8, p. 160], from the last correlation we will get the inequality

$$\mathcal{E}\left(W_\infty^r;U_\delta(f;\Lambda)\right)_C\leq U_\delta\left(\psi_r;0;\Lambda\right)-\psi_r(0),$$

where

$$\psi_r(t)=\varphi_r\left(t+\frac{\pi}{2}(r-1)\right).$$

As $\varphi_r\in W_\infty^r$, so

$$\mathcal{E}\left(W_\infty^r;U_\delta(f;\Lambda)\right)_C=U_\delta\left(\psi_r;0;\Lambda\right)-\psi_r(0).$$

As our next reflections will be identical to the proof of theorem 3 of the work [6], then analogically examining the function

$$f_r(t)=\frac{1}{4}\psi_{r-1}(t),$$

functional $g$, which is determined with help of function $-\operatorname{sign}f_r(t)$, and operator $A_gf_r$, which is determined as

$$A_gf_r=g\left(f_r(s+t)\right)=-\int\limits_0^{2\pi}f_r(s+t)\operatorname{sign}f_r(s)ds,$$

we will get

$$U_\delta\left(A_gf_r;\Lambda;0\right)-A_gf_r(0)=\mathcal{E}\left(W_\infty^r;U_\delta(f;\Lambda)\right)_C.$$

As $f_r^{r-1}(t)$ is a maximum point of the plurality $W_1^1$, so

$$\mathcal{E}\left(W_1^r;U_\delta(f;\Lambda)\right)_1=\mathcal{E}\left(W_\infty^r;U_\delta(f;\Lambda)\right)_C.$$

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COROLLARY 2.2. *If the succession* $\{\lambda_\delta(k)\}_{k=\overline{0,\infty}}$ *is concave downward, so* $\Delta^2\lambda_\delta(k)\geq 0$, *and monotonously goes to zero, then according to* [9, pp.297–298] *the equality* (5) *take place.*

Applying the proved above theorems from now on it is possible for many linear methods of summing up, which are set up by the succession of functions of natural argument, which is dependent on some real parameter and having the results, which are got for the case of uniform metric, to get analogical results in the integral metric too.

Let $f \in L$. We will examine as an example the succession $\Lambda = \{\lambda_\delta(k)\}$ that

$$\lambda_\delta(k) = \left[1 + \frac{k}{2}\left(1 - e^{-\frac{2}{\delta}}\right)\right] e^{-\frac{k}{\delta}}, \ \delta > 0. \tag{6}$$

Then according to the equality (1) we will get an expression

$$B_\delta(f, x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t + x)K_\delta(t)dt, \tag{7}$$

which is usually called the biharmonic integral of Poisson of the function $f(x)$, where

$$K_\delta(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}\left(1 - e^{-\frac{2}{\delta}}\right)\right] e^{-\frac{k}{\delta}} \cos kt \tag{8}$$

is a biharmonic kernel of Poisson.

Now let $\delta = -\frac{1}{\ln \rho}$, $0 \le \rho < 1$. Then, biharmonic integral we can write down in the form

$$B_\rho(f, x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t + x)\left(\frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}(1 - \rho^2)\right]\rho^k \cos kt\right)dt. \tag{9}$$

Let's give the definition, which is necessary for us for the next exposition of the material.

Definition 2.3. The formal series $\sum\limits_{n=0}^{\infty} g_n(\rho)$ we will call *the full asymptotic decomposition* of the function $f(\rho)$ when $\rho \to 1-$, if for every $n \in \mathbb{N}$

$$|g_{n+1}(\rho)| = o(|g_n(\rho)|) \tag{10}$$

and for an arbitrary $m \in \mathbb{N}$

$$f(\rho) = \sum_{n=0}^{m} g_n(\rho) + o(g_m(\rho)), \ rho \to 1-. \tag{11}$$

In short this fact can be written in the following way: $f(\delta) \cong \sum\limits_{n=0}^{\infty} g_n(\rho)$.

In work [10] the full asymptotic decomposition for powers $(1 - \rho)$ when $\rho \to 1-$ was got for the quantities

$$\mathcal{E}(W_\infty^r, B_\rho)_C = \sup_{f \in W_\infty^r} \|f(x) - B_\rho(f, x)\|_C. \tag{12}$$

Using the proved by us theorem we can write down the analogical decompositions for the case of integral metric.

As for biharmonic integral of Poisson the conditions of theorem 2 are true, so the next theorems will be valid as well:

THEOREM 2.4. *The full asymptotic decomposition takes place:*

$$\mathcal{E}\left(W_1^1, B_\rho\right)_1 \cong \frac{2}{\pi}(1-\rho) + \frac{2}{\pi}(1-\rho)^2 \ln\frac{1}{1-\rho} + \frac{2\ln 2+1}{\pi}(1-\rho)^2 +$$

$$+\frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\frac{1}{k}(1-\rho)^k \ln\frac{1}{1-\rho} + \gamma_k (1-\rho)^k\right\},$$

$$\gamma_k = \frac{1}{k}\Big(\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\Big) - \frac{1}{(k-2)(k-1)2^{k-1}}.$$

THEOREM 2.5. *If* $r = 2l+1$, $l \in N$, *then, when* $\rho \to 1-$, *the full asymptotic decomposition takes place:*

$$\mathcal{E}\left(W_1^r, B_\rho\right)_1 \cong \left(\tilde{K}_{r-1} + \frac{1}{2}K_{r-2}\right)(1-\rho)^2 +$$

$$+\frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\left[\alpha_k^r + \alpha_{k-1}^{r-1} - \frac{1}{2}\alpha_{k-2}^{r-1}\right](1-\rho)^k \ln\frac{1}{1-\rho} + \left[\beta_k^r + \beta_{k-1}^{r-1} - \frac{1}{2}\beta_{k-2}^{r-1}\right](1-\rho)^k\right\},$$

*in which*

$$\alpha_k^n = \frac{(-1)^k}{k!}a_n^k,$$

$$\beta_k^n = \frac{(-1)^k}{k!}\left\{\sum_{i=1}^{n-1}\varphi_{n-i}(0)\, a_i^k + a_n^k\left(\ln 2 + \sum_{i=1}^{k}\frac{1}{i}\right) + S_k^n\right\},$$

$$\varphi_n(0) = \begin{cases} \frac{\pi}{2}K_n, & n = 2l-1, \\ \frac{\pi}{2}\tilde{K}_n, & n = 2l, \end{cases} \quad l \in N,$$

*where* $K_n$ *and* $\tilde{K}_n$ *are well-known constants of J. Favard–N. I. Akhiezer–M. G. Krein*

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}},\ n = 0,1,2,...,\quad \tilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{mn}}{(2m+1)^{n+1}},\ n \in N.$$

$$S_k^n = \begin{cases} 0, k \le n, \\ \sum\limits_{i=n+1}^{k}\frac{a_i^k}{2^{i-n}} + \sum\limits_{i=1}^{k-n}A_i^{k-1}a_n^{k-i}, k > n, \end{cases}$$

$$a_i^j = \begin{cases} 0, i > j, \\ (-1)^j (j-1)!, i = 1, \\ a_{i-1}^{j-1} - a_i^{j-1}(j-1), i \le j \le n, \\ a_{i-1}^{j-1} - a_i^{j-1}(j-2), n+1 = i \le j, \\ -(i-n-1)\, a_{i-1}^{j-1} - a_i^{j-1}(j-i+n-1), n+1 < i \le j, \end{cases}$$

$$A_k^n = (-1)^{k-1}\frac{n(n-1)...(n-k+1)}{k}.$$

THEOREM 2.6. *If* $r = 2l$, $l \in N$, *then, when* $\rho \to 1-$, *the full asymptotic decomposition takes place:*

$$\mathcal{E}\left(W^r, B_\rho\right)_1 \cong \left(\tilde{K}_{r-1} + \frac{1}{2}K_{r-2}\right)(1-\rho)^2 + \frac{4}{\pi}\sum_{k=3}^{\infty}\left[\gamma_k^r + \gamma_{k-1}^{r-1} - \frac{1}{2}\gamma_{k-2}^{r-1}\right](1-\rho)^k,$$

*in which*

$$\gamma_k^n = \frac{(-1)^k}{k!}\left\{\sum_{i=1}^{n} \psi_{n-i}(0)\, b_i^k + \sigma_k^n\right\},$$

$$\psi_n(0) = \begin{cases} \frac{\pi}{4}K_n, & n = 2l, \\ \frac{\pi}{4}\tilde{K}_n, & n = 2l-1, \end{cases} \quad l \in N,$$

$$\sigma_k^n = \begin{cases} 0, & k \le n, \\ \sum\limits_{i=n+1}^{k} \frac{b_i^k}{2^{i-n}}, & k > n, \end{cases}$$

$$b_i^j = \begin{cases} 0, \quad i > j, \\ (-1)^j (j-1)!, \quad i = 1, \\ b_{i-1}^{j-1} - b_i^{j-1}(j-1), \quad i \le j \le n, \\ b_{i-1}^{j-1} - b_i^{j-1}(j-2), \quad n+1 = i \le j, \\ -2(i-n-1)\, b_{i-1}^{j-1} - b_i^{j-1}(j-2i+2n), \quad n+1 < i \le j. \end{cases}$$

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