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APPROXIMATION OF (ψ, β)-DIFFERENTIABLE FUNCTIONS DEFINED ON THE REAL AXIS BY WEIERSTRASS OPERATORS

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We obtain asymptotic equalities for the upper bounds of approximations by Weierstrass operators on the functional classes $\hat{C}^{\psi}_{\beta,\infty}$ and $\hat{L}^{\psi}_{\beta,1}$ in the metrics of the spaces $\hat{C}$ and $\hat{L}_1$, respectively.

1. Main Definitions

Let $\hat{L}_p$, $p \geq 1$, be the set of functions $f(\cdot)$ defined on the entire real axis R and having the finite norm

$$\|f\|_{\hat{p}} = \begin{cases} \sup\limits_{a\in R} \left(\int_a^{a+2\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}}, & p \in [1, \infty), \\ \operatorname*{ess\,sup}\limits_{t\in R} |f(t)|, & p = \infty, \end{cases}$$

and let $\widehat{C}$ be the set of continuous functions defined on the real axis and having the finite norm

$$\|f\|_{\hat{C}} = \max_{t \in R} |f(t)|.$$

Let $\mathfrak{A}$ be the set of positive functions $\psi(t)$ continuous for $t \geq 0$ and satisfying the following conditions:

(i) $\psi(0) = 0$;

(ii) $\psi(t)$ is convex downwards on $[1, \infty)$ and $\lim\limits_{t\to\infty} \psi(t) = 0$;

(iii) $\psi'(t) = \psi'(t+0)$ is a function of bounded variation on $[0, \infty)$.

The subset of functions $\psi \in \mathfrak{A}$ for which

$$\int_1^{\infty} \frac{\psi(t)}{t} dt < \infty$$

is denoted by $\mathfrak{A}'$.

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If $\psi \in \mathfrak{A}'$ and $\beta \in R,$ then the transform

$$\widehat{\psi}_\beta(t) = \widehat{\psi}\,(t;\beta) = \frac{1}{\pi}\int\limits_0^\infty \psi(v)\cos\left(vt + \frac{\beta\pi}{2}\right)dv \tag{1}$$

is known to be summable on the entire real axis (see [1, p. 194]).

Let $\widehat{L}^\psi_\beta$ denote the set of functions $f \in \widehat{L}_1$ that can be represented in the following form for almost all $x \in R$:

$$f(x) = A_0 + \int\limits_{-\infty}^{\infty} \varphi\,(x+t)\,\widehat{\psi}_\beta(t)dt = A_0 + \left(\varphi * \widehat{\psi}_\beta\right)(x), \tag{2}$$

where A_0 is a certain constant, $\varphi \in \widehat{L}_1,$ and the integral is understood as the limit of integrals taken over increasing symmetric segments (see, e.g., [2, 3]). If $f \in \widehat{L}^\psi_\beta$ and $\varphi \in \mathfrak{N},$ where $\mathfrak{N}$ is a certain subset of $\widehat{L}_1,$ then one sets $f \in \widehat{L}^\psi_\beta\mathfrak{N}.$ The subsets of continuous functions from $\widehat{L}^\psi_\beta$ and $\widehat{L}^\psi_\beta\mathfrak{N}$ are denoted by $\widehat{C}^\psi_\beta$ and $\widehat{C}^\psi_\beta\mathfrak{N},$ respectively. The function $\varphi(\cdot)$ in (2) is called the (ψ, β)-derivative of the function $f(\cdot)$ and is denoted by $f^\psi_\beta(\cdot).$

Let $\widehat{C}^\psi_{\beta,\infty}$ denote the set of functions $f \in \widehat{C}^\psi_\beta\mathfrak{N}$ in the case where $\mathfrak{N}$ coincides with the unit ball in the space $\widehat{L}_\infty,$ i.e.,

$$\mathfrak{N} = S_\infty = \left\{\varphi \in \widehat{L}_\infty \colon \operatorname*{ess\,sup}_{t\in R} |\varphi(t)| \le 1\right\}$$

and let $\widehat{L}^\psi_{\beta,1}$ denote the set of functions $f \in \widehat{L}^\psi_\beta\mathfrak{N}$ in the case where $\mathfrak{N}$ is the unit ball in the space $\widehat{L}_1,$ i.e.,

$$\mathfrak{N} = S_1 = \left\{\varphi \in \widehat{L}_1 \colon \|\varphi\|_{\hat{1}} \le 1\right\}.$$

It was shown in [1, p. 169] that, in the case where $\varphi(\cdot)$ is a 2π-periodic function such that

$$\int\limits_{-\pi}^{\pi} \varphi(t)\,dt = 0,$$

the classes $\widehat{L}^\psi_\beta\mathfrak{N},$ $\widehat{L}^\psi_{\beta,1},$ and $\widehat{C}^\psi_{\beta,\infty}$ turn into the known classes $L^\psi_\beta\mathfrak{N},$ $L^\psi_{\beta,1},$ and $C^\psi_{\beta,\infty},$ respectively.

Further, let

$$\mathfrak{A}_0 := \left\{\psi \in \mathfrak{A} \colon 0 < \frac{t}{\eta(t) - t} \le K < \infty \quad \forall t \ge 1\right\},$$

where

$$\eta(t) = \eta(\psi, t) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right)$$

and ψ^{-1} is the function inverse to $\psi.$

Now consider the collection of functions $\Lambda = \left\{\lambda_\sigma\left(\frac{v}{\sigma}\right)\right\}$ continuous for $v \geq 0$ and dependent on a real parameter σ. We associate every function $f \in \hat{L}^\psi_\beta$ with the expression

$$U_\sigma(f;x;\Lambda) = A_0 + \int\limits_{-\infty}^{\infty} f^\psi_\beta\,(x+t)\,\frac{1}{\pi}\int\limits_0^{\infty}\psi(v)\lambda_\sigma\left(\frac{v}{\sigma}\right)\cos\left(vt+\frac{\beta\pi}{2}\right)dvdt. \qquad (3)$$

We approximate functions from the classes $\widehat{L}^\psi_{\beta,1}$ and $\widehat{C}^\psi_{\beta,\infty}$ by operators of the form (3) in the case where $\lambda_\sigma\left(\frac{v}{\sigma}\right) = e^{-\frac{v^2}{\sigma}}$. We denote these operators by $W_\sigma(f;x)$, $\sigma \in (0,\infty)$, and call them Weierstrass operators:

$$W_\sigma(f;x) = A_0 + \int\limits_{-\infty}^{\infty} f^\psi_\beta(x+t)\frac{1}{\pi}\int\limits_0^{\infty}\psi(v)e^{-\frac{v^2}{\sigma}}\cos\left(vt+\frac{\beta\pi}{2}\right)dv\,dt, \quad \sigma\in(0,\infty). \qquad (4)$$

Using Proposition 1.1 from [1, p. 169], one can easily verify that, for a 2π-periodic function f, the operators $W_\sigma(f;x)$ coincide with the known Weierstrass integrals

$$W_\sigma(f;x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^2}{\sigma}}\big(a_k\cos kx + b_k\sin kx\big), \quad \sigma > 0$$

(see. e.g., [4, p. 150]).

In the present paper, we investigate the asymptotic behavior of the quantities

$$\mathcal{E}\left(\widehat{C}^\psi_{\beta,\infty};W_\sigma\right)_{\widehat{C}} = \sup_{f\in\widehat{C}^\psi_{\beta,\infty}}\|f(x)-W_\sigma(f;x)\|_{\widehat{C}},$$

$$\mathcal{E}\left(\widehat{L}^\psi_{\beta,1};W_\sigma\right)_{\widehat{1}} = \sup_{f\in\widehat{L}^\psi_{\beta,1}}\|f(x)-W_\sigma(f;x)\|_{\widehat{1}}$$

as $\sigma\to\infty$.

If a function $h(\sigma) = h(\mathfrak{N};\sigma)$ such that

$$\mathcal{E}\left(\mathfrak{N};W_\sigma\right)_X = h\left(\sigma\right) + o\left(h\left(\sigma\right)\right) \quad \text{for} \quad \sigma\to\infty$$

is found in explicit form, then one says that the Nikol'skii–Kolmogorov problem is solved for a Weierstrass operator on the class $\mathfrak{N}$ in the metric of the space X.

Note that the Nikol'skii–Kolmogorov problem for Weierstrass integrals on the classes W^r_β and W^r, the Zygmund classes, and other classes was solved by Korovkin [5], Bausov [6, 7], Bugrov [8], Baskakov [9], and Falaleev [10]. The results of the present paper are closely related to the results obtained by Kharkevych and Kal'chuk in [11], where the Nikol'skii–Kolmogorov problem was solved for Weierstrass integrals on the classes $C^\psi_{\beta,\infty}$ and $L^\psi_{\beta,1}$.

2. Estimation of Upper Bounds of Functions on the Classes $\widehat{C}^{\psi}_{\beta,\infty}$ by Their Weierstrass Operators

We set

$$\tau(v) = \tau_\sigma(v, \psi) = \left(1 - e^{-v^2}\right) \frac{\psi(\sqrt{\sigma} v)}{\psi(\sqrt{\sigma})}, \quad v \geq 0, \tag{5}$$

where $\psi(v)$ is a function defined and continuous for all $v \geq 0$. In what follows, we assume that the function $\psi(v)$ is monotonically increasing and convex downwards on $[0, 1]$ and has the continuous second derivative for all $v \geq 0$ except the point $v = 1$. Denote the sets of functions $\psi \in \mathfrak{A}$ and $\psi \in \mathfrak{A}_0$ with the properties indicated above by $\mathfrak{A}^*$ and $\mathfrak{A}_0^*$, respectively.

Theorem 1. *Suppose that $\psi \in \mathfrak{A}_0^* \cap \mathfrak{A}'$ and the function $g(v) = v^2 \psi(v)$ is convex upwards or downwards on $[b, \infty)$, $b \geq 1$. Then the following equality holds for $\sigma \to \infty$:*

$$\mathcal{E}\left(\widehat{C}^{\psi}_{\beta,\infty}; W_\sigma\right)_{\widehat{C}} = \psi(\sqrt{\sigma}) A(\tau), \tag{6}$$

where $A(\tau)$ is defined by the relation

$$A(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \int_0^{\infty} \tau(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt \tag{7}$$

and satisfies the estimate

$$A(\tau) = \frac{2}{\pi} \left|\sin \frac{\beta\pi}{2}\right| \left(\frac{1}{\sigma \psi(\sqrt{\sigma})} \int_1^{\sqrt{\sigma}} v \psi(v) dv + \frac{1}{\psi(\sqrt{\sigma})} \int_{\sqrt{\sigma}}^{\infty} \frac{\psi(v)}{v} dv \right) + O\left(1 + \frac{1}{\sigma \psi(\sqrt{\sigma})}\right). \tag{8}$$

Proof. It follows from Lemma 1 in [12] that, to prove equality (6), it suffices to show the summability of a transform $\widehat{\tau}_\beta(t)$ of the function $\tau(v)$ of the form

$$\widehat{\tau}_\beta(t) = \widehat{\tau}(t, \beta) = \frac{1}{\pi} \int_0^{\infty} \tau(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv, \tag{9}$$

i.e., to establish the convergence of integral (7).

According to Theorem 1 in [7, p. 24], for the convergence of integral (7) it is necessary and sufficient that the following integrals be convergent:

$$\int_0^{\frac{1}{2}} v \left|d\tau'(v)\right|, \qquad \int_{\frac{1}{2}}^{\infty} |v - 1| \left|d\tau'(v)\right|, \tag{10}$$

$$\left|\sin \frac{\beta\pi}{2}\right| \int_0^{\infty} \frac{|\tau(v)|}{v} dv, \qquad \int_0^1 \frac{|\tau(1 - v) - \tau(1 + v)|}{v} dv. \tag{11}$$

Let us estimate the first integral in (10). To this end, we divide the segment of integration into two parts: $\left[0, \dfrac{1}{\sqrt{\sigma}}\right]$ and $\left[\dfrac{1}{\sqrt{\sigma}}, \dfrac{1}{2}\right]$ (for $\sigma > 4b^2$).

Taking into account that $\tau''(v) \geq 0$ on $\left[0, \dfrac{1}{\sqrt{\sigma}}\right]$ and using the inequality

$$1 - e^{-v^2} \leq v^2, \quad v \in R, \tag{12}$$

we get

$$\int\limits_0^{\frac{1}{\sqrt{\sigma}}} v \left|d\tau'(v)\right| = \big(v\tau'(v) - \tau(v)\big)\Big|_0^{\frac{1}{\sqrt{\sigma}}}$$

$$= \frac{\psi'(1-0)}{\psi\left(\sqrt{\sigma}\right)}(1 - e^{-\frac{1}{\sigma}}) + \frac{2\psi(1)}{\psi\left(\sqrt{\sigma}\right)} \frac{1}{\sigma} e^{-\frac{1}{\sigma}} - \frac{\psi(1)}{\psi(\sqrt{\sigma})}\left(1 - e^{-\frac{1}{\sigma}}\right)$$

$$= O\left(\frac{1}{\sigma\psi\left(\sqrt{\sigma}\right)}\right). \tag{13}$$

By virtue of equalities (21) and (27)–(29) from [11], the following estimate is true:

$$\int\limits_{\frac{1}{\sqrt{\sigma}}}^{\frac{1}{2}} v \left|d\tau'(v)\right| = O\left(1 + \frac{1}{\sigma\psi(\sqrt{\sigma})}\right), \quad \sigma \to \infty. \tag{14}$$

Combining relations (13) and (14), we get

$$\int\limits_0^{\frac{1}{2}} v \left|d\tau'(v)\right| = O\left(1 + \frac{1}{\sigma\psi(\sqrt{\sigma})}\right), \quad \sigma \to \infty. \tag{15}$$

Taking into account equality (34) from [11], we conclude that the following estimate holds for the second integral in (10):

$$\int\limits_{\frac{1}{2}}^{\infty} |v - 1| \left|d\tau'(v)\right| = O(1). \tag{16}$$

To estimate the first integral in (11), we divide the interval $[0, \infty)$ into two parts: $\left[0, \dfrac{1}{\sqrt{\sigma}}\right]$ and $\left[\dfrac{1}{\sqrt{\sigma}}, \infty\right)$.

Let $v\in\left[0,\frac{1}{\sqrt{\sigma}}\right]$. Taking into account inequality (12) and the fact that the function $\psi(v)$ is monotonically increasing on $[0,1]$, we obtain

$$\int\limits_0^{\frac{1}{\sqrt{\sigma}}}\frac{|\tau(v)|}{v}dv=\frac{1}{\psi\left(\sqrt{\sigma}\right)}\int\limits_0^{\frac{1}{\sqrt{\sigma}}}\left(1-e^{-v^2}\right)\psi\left(\sqrt{\sigma}v\right)\frac{dv}{v}\le\frac{\psi(1)}{\psi\left(\sqrt{\sigma}\right)}\int\limits_0^{\frac{1}{\sqrt{\sigma}}}vdv=O\left(\frac{1}{\sigma\psi\left(\sqrt{\sigma}\right)}\right). \tag{17}$$

Using estimates (36) and (37) from [11], we get

$$\int\limits_{\frac{1}{\sqrt{\sigma}}}^{\infty}\frac{|\tau(v)|}{v}dv=\frac{1}{\sigma\psi(\sqrt{\sigma})}\int\limits_1^{\sqrt{\sigma}}v\psi(v)dv+\frac{1}{\psi(\sqrt{\sigma})}\int\limits_{\sqrt{\sigma}}^{\infty}\frac{\psi(v)}{v}dv+O\left(1+\frac{1}{\sigma\psi(\sqrt{\sigma})}\right). \tag{18}$$

Combining equalities (17) and (18), we conclude that the following equality holds for the first integral in (11):

$$\int\limits_0^{\infty}\frac{|\tau(v)|}{v}dv=\frac{1}{\sigma\psi(\sqrt{\sigma})}\int\limits_1^{\sqrt{\sigma}}v\psi(v)dv+\frac{1}{\psi(\sqrt{\sigma})}\int\limits_{\sqrt{\sigma}}^{\infty}\frac{\psi(v)}{v}dv+O\left(1+\frac{1}{\sigma\psi(\sqrt{\sigma})}\right). \tag{19}$$

Let us show that the following estimate holds for the second integral:

$$\int\limits_0^1\frac{|\tau(1-v)-\tau(1+v)|}{v}dv=O\left(1+\frac{1}{\sigma\psi(\sqrt{\sigma})}\right),\quad \sigma\to\infty. \tag{20}$$

Using relation (5), we obtain

$$\tau(1-v)=\left(1-e^{-(1-v)^2}\right)\frac{\psi\left(\sqrt{\sigma}(1-v)\right)}{\psi(\sqrt{\sigma})},\quad v\le 1, \tag{21}$$

$$\tau(1+v)=\left(1-e^{-(1+v)^2}\right)\frac{\psi\left(\sqrt{\sigma}(1+v)\right)}{\psi(\sqrt{\sigma})},\quad v\ge -1. \tag{22}$$

We represent the integral on the left-hand side of equality (20) as a sum of two integrals:

$$\int\limits_0^1\frac{|\tau(1-v)-\tau(1+v)|}{v}dv=\left(\int\limits_0^{1-\frac{1}{\sqrt{\sigma}}}+\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^1\right)\frac{|\tau(1-v)-\tau(1+v)|}{v}dv. \tag{23}$$

Let us estimate the first term on the right-hand side of equality (23). To this end, we subtract and add the expression $e^{-(1-v)^2}-e^{-(1+v)^2}$ under the modulus sign in the integrand. As a result, we get

$$\int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{|\tau(1-v)-\tau(1+v)|}{v}dv = O\left(\int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{\left|e^{-(1-v)^2}-e^{-(1+v)^2}\right|}{v}dv\right.$$

$$\left.+\int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{\left|\tau(1-v)-\tau(1+v)+e^{-(1-v)^2}-e^{-(1+v)^2}\right|}{v}dv\right). \qquad (24)$$

It is obvious that

$$\int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{\left|e^{-(1-v)^2}-e^{-(1+v)^2}\right|}{v}dv = O(1). \qquad (25)$$

We now estimate the second integral on the right-hand side of (24). By virtue of relations (21) and (22), we have

$$e^{-(1-v)^2} = 1 - \frac{\psi(\sqrt{\sigma})}{\psi(\sqrt{\sigma}(1-v))}\tau(1-v), \quad v \le 1, \qquad (26)$$

$$e^{-(1+v)^2} = 1 - \frac{\psi(\sqrt{\sigma})}{\psi(\sqrt{\sigma}(1+v))}\tau(1+v), \quad v \ge -1. \qquad (27)$$

We obtain

$$\int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{\left|\tau(1-v)-\tau(1+v)+e^{-(1-v)^2}-e^{-(1+v)^2}\right|}{v}dv$$

$$\le \int_0^{1-\frac{1}{\sqrt{\sigma}}} |\tau(1-v)|\left|1-\frac{\psi(\sqrt{\sigma})}{\psi(\sqrt{\sigma}(1-v))}\right|\frac{dv}{v} + \int_0^{1-\frac{1}{\sqrt{\sigma}}} |\tau(1+v)|\left|1-\frac{\psi(\sqrt{\sigma})}{\psi(\sqrt{\sigma}(1+v))}\right|\frac{dv}{v}. \qquad (28)$$

Using relations (15) and (16) and Lemma 2 in [7, p. 19], we get

$$\int_0^{1-\frac{1}{\sqrt{\sigma}}} |\tau(1-v)|\left|1-\frac{\psi(\sqrt{\sigma})}{\psi(\sqrt{\sigma}(1-v))}\right|\frac{dv}{v} + \int_0^{1-\frac{1}{\sqrt{\sigma}}} |\tau(1+v)|\left|1-\frac{\psi(\sqrt{\sigma})}{\psi(\sqrt{\sigma}(1+v))}\right|\frac{dv}{v}$$

$$= H(\tau)O\left(\int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{|\psi(\sqrt{\sigma}(1-v))-\psi(\sqrt{\sigma})|}{v\psi(\sqrt{\sigma}(1-v))}dv + \int_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{|\psi(\sqrt{\sigma}(1+v))-\psi(\sqrt{\sigma})|}{v\psi(\sqrt{\sigma}(1+v))}dv\right), \qquad (29)$$

where

$$H(\tau) = |\tau(0)| + |\tau(1)| + \int\limits_0^{\frac{1}{2}} v \left| d\tau'(v) \right| + \int\limits_{\frac{1}{2}}^{\infty} |v-1| \left| d\tau'(v) \right|. \tag{30}$$

Let us show that, for $\sigma \to \infty,$ we have

$$I_{1,\sigma} := \int\limits_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{|\psi(\sqrt{\sigma}(1-v)) - \psi(\sqrt{\sigma})|}{v\psi(\sqrt{\sigma}(1-v))} dv = O(1), \tag{31}$$

$$I_{2,\sigma} := \int\limits_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{|\psi(\sqrt{\sigma}(1+v)) - \psi(\sqrt{\sigma})|}{v\psi(\sqrt{\sigma}(1+v))} dv = O(1), \tag{32}$$

where $O(1)$ is uniformly bounded in σ.

Indeed, the function $\dfrac{1 - \psi(\sqrt{\sigma})/\psi(\sqrt{\sigma}(1-v))}{v}$ is bounded for all $v \in \left[\theta,\ 1 - \dfrac{1}{\sqrt{\sigma}}\right],\ 0 < \theta < 1 - \dfrac{1}{\sqrt{\sigma}},$ and, furthermore, in view of Theorem 3.12.1 in [13, p. 161], we have

$$\lim_{v \to 0} \frac{1 - \psi(\sqrt{\sigma})/\psi(\sqrt{\sigma}(1-v))}{v} = \frac{\sqrt{\sigma}\,|\psi'(\sqrt{\sigma})|}{\psi(\sqrt{\sigma})} \le K.$$

Thus, $I_{1,\sigma} = O(1)$ for $\sigma \to \infty.$

Passing to the estimation of the integral $I_{2,\sigma},$ we note that

$$I_{2,\sigma} < \frac{1}{\psi(2\sqrt{\sigma}-1)} \int\limits_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{\psi(\sqrt{\sigma}) - \psi\left(\sqrt{\sigma}\,(1+v)\right)}{v} dv.$$

Changing the variable $u = \sqrt{\sigma}(1+v),$ we get

$$I_{2,\sigma} < \frac{1}{\psi(2\sqrt{\sigma}-1)} \int\limits_{\sqrt{\sigma}}^{2\sqrt{\sigma}-1} \frac{\psi(\sqrt{\sigma}) - \psi\left(u\right)}{u - \sqrt{\sigma}} du < \frac{1}{\psi(2\sqrt{\sigma}-1)} \int\limits_{\sqrt{\sigma}}^{2\sqrt{\sigma}} \frac{\psi(\sqrt{\sigma}) - \psi\left(u\right)}{u - \sqrt{\sigma}} du.$$

Applying Lemma 3.5.5 from [14, p. 97] and Theorem 3.16.1 from [13, p. 175] to the right-hand side of the last inequality, we obtain

$$I_{2,\sigma} < \frac{K_1\psi(\sqrt{\sigma})}{\psi(2\sqrt{\sigma}-1)} \le \frac{K_2\psi(\sqrt{\sigma})}{\psi\left(2\sqrt{\sigma}\right)} \le K_3.$$

Thus, equalities (31) and (32) are true.

Combining relations (28)–(32), we get

$$\int\limits_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{\left|\tau(1-v)-\tau(1+v)+e^{-(1-v)^2}-e^{-(1+v)^2}\right|}{v}dv = H(\tau)O(1). \tag{33}$$

According to (15) and (16), the following estimate holds for the quantity $H(\tau)$ defined by (30):

$$H(\tau) = O\left(1+\frac{1}{\sigma\psi(\sqrt{\sigma})}\right). \tag{34}$$

Using relations (24), (25), (33), and (34), we obtain

$$\int\limits_0^{1-\frac{1}{\sqrt{\sigma}}} \frac{|\tau(1-v)-\tau(1+v)|}{v}dv = O\left(1+\frac{1}{\sigma\psi(\sqrt{\sigma})}\right). \tag{35}$$

Let us estimate the second term on the right-hand side of (23). To this end, we subtract and add the quantity

$$\frac{\psi(\sqrt{\sigma}(1-v))}{\psi(1)}\left(e^{-(1-v)^2}-e^{-(1+v)^2}\right)$$

under the modulus sign of the integrand and take into account that the function $\psi(\sqrt{\sigma}(1-v))$ is monotonically decreasing on $\left[1-\dfrac{1}{\sqrt{\sigma}},1\right]$. As a result, we get

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{|\tau(1-v)-\tau(1+v)|}{v}dv$$

$$\leq \frac{1}{\psi(1)}\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{\psi(\sqrt{\sigma}(1-v))\left|e^{-(1-v)^2}-e^{-(1+v)^2}\right|}{v}dv$$

$$+\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{\left|\tau(1-v)-\tau(1+v)+\frac{\psi(\sqrt{\sigma}(1-v))}{\psi(1)}\left(e^{-(1-v)^2}-e^{-(1+v)^2}\right)\right|}{v}dv$$

$$\leq \int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{\left|e^{-(1-v)^2} - e^{-(1+v)^2}\right|}{v} dv$$

$$+ \int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{\left|\tau(1-v) - \tau(1+v) + \frac{\psi\left(\sqrt{\sigma}(1-v)\right)}{\psi(1)}\left(e^{-(1-v)^2} - e^{-(1+v)^2}\right)\right|}{v} dv. \tag{36}$$

It is obvious that

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{\left|e^{-(1-v)^2} - e^{-(1+v)^2}\right|}{v} dv = O(1). \tag{37}$$

Using relations (26) and (27) and Lemma 2 from [7, p. 19], we get

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{\left|\tau(1-v) - \tau(1+v) + \frac{\psi\left(\sqrt{\sigma}(1-v)\right)}{\psi(1)}\left(e^{-(1-v)^2} - e^{-(1+v)^2}\right)\right|}{v} dv$$

$$= \int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \frac{1}{v}\Bigg|\tau(1-v) - \tau(1+v) + \frac{\psi\left(\sqrt{\sigma}(1-v)\right)}{\psi(1)}$$

$$\times \left(\frac{\psi(\sqrt{\sigma})}{\psi\left(\sqrt{\sigma}(1-v)\right)}\tau(1-v) + \frac{\psi\left(\sqrt{\sigma}\right)}{\psi\left(\sqrt{\sigma}(1+v)\right)}\tau(1+v)\right)\Bigg| dv$$

$$\leq \int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} |\tau(1-v)| \left|1 - \frac{\psi(\sqrt{\sigma})}{\psi(1)}\right| \frac{dv}{v} + \int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} |\tau(1+v)| \left|1 - \frac{\psi\left(\sqrt{\sigma}(1-v)\right)\psi(\sqrt{\sigma})}{\psi(1)\psi\left(\sqrt{\sigma}(1+v)\right)}\right| \frac{dv}{v}$$

$$= H(\tau) O\left(\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \left|1 - \frac{\psi(\sqrt{\sigma})}{\psi(1)}\right| \frac{dv}{v} + \int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \left|1 - \frac{\psi\left(\sqrt{\sigma}(1-v)\right)\psi(\sqrt{\sigma})}{\psi(1)\psi\left(\sqrt{\sigma}(1+v)\right)}\right| \frac{dv}{v}\right), \tag{38}$$

where $H(\tau)$ is defined by (30).

We estimate the first integral on the right-hand side of (38) as follows:

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1} \left|1 - \frac{\psi(\sqrt{\sigma})}{\psi(1)}\right| \frac{dv}{v} = \left(1 - \frac{\psi(\sqrt{\sigma})}{\psi(1)}\right) \ln \frac{1}{1 - \frac{1}{\sqrt{\sigma}}} = O(1). \tag{39}$$

Since the function $\left|1-\frac{\psi\left(\sqrt{\sigma}(1-v)\right)\psi(\sqrt{\sigma})}{\psi(1)\psi\left(\sqrt{\sigma}(1+v)\right)}\right|$ is bounded on $\left[1-\frac{1}{\sqrt{\sigma}},1\right]$, we get

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1}\left|1-\frac{\psi\left(\sqrt{\sigma}(1-v)\right)\psi(\sqrt{\sigma})}{\psi(1)\psi\left(\sqrt{\sigma}(1+v)\right)}\right|\frac{dv}{v}=O(1). \tag{40}$$

Using relations (38)–(40), we obtain

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1}\frac{\left|\tau(1-v)-\tau(1+v)+\frac{\psi\left(\sqrt{\sigma}(1-v)\right)}{\psi(1)}\left(e^{-(1-v)^2}-e^{-(1+v)^2}\right)\right|}{v}dv=H(\tau)O(1). \tag{41}$$

Taking into account relations (36), (37), and (41) and estimate (34), we get

$$\int\limits_{1-\frac{1}{\sqrt{\sigma}}}^{1}\frac{|\tau(1-v)-\tau(1+v)|}{v}dv=O\left(1+\frac{1}{\sigma\psi(\sqrt{\sigma})}\right). \tag{42}$$

Combining relations (35) and (42), we arrive at equality (20).

Using inequalities (2.14) and (2.15) from [7, p. 24] and relations (19), (20), and (34), we obtain relation (8).

Theorem 1 is proved.

Theorem 1 yields the following statements:

Corollary 1. *If the conditions of Theorem 1 are satisfied,* $\sin\frac{\beta\pi}{2}\neq 0,$ *and* $\lim\limits_{t\to\infty}\alpha(t)=\infty,$ *where*

$$\alpha(t)=\frac{\psi(t)}{t\,|\psi'(t)|}, \tag{43}$$

then the following asymptotic equality holds as $\sigma\to\infty$:

$$\mathcal{E}\left(\widehat{C}^{\psi}_{\beta,\infty};W_\sigma\right)_{\widehat{C}}=\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int\limits_{\sqrt{\sigma}}^{\infty}\frac{\psi(v)}{v}dv+O\left(\psi(\sqrt{\sigma})\right). \tag{44}$$

Examples of functions that satisfy the conditions of Corollary 1 are the functions $\psi\in\mathfrak{A}^*$ that have the following form on the interval $[1,\infty)$:

$$\psi(v)=\frac{1}{\ln^{\alpha}(v+K)},$$

where $\alpha>1$ and $K>0$.

Corollary 2. *Suppose that* $\psi \in \mathfrak{A}_0^*$, $\sin\dfrac{\beta\pi}{2} \neq 0$, *the function* $v^2\psi(v)$ *is convex upwards or downwards on the interval* $[b,\infty)$, $b \geq 1$, *and*

$$\lim_{v\to\infty} v^2\psi(v) = \infty,$$

$$\lim_{\sigma\to\infty} \frac{1}{\sigma\psi(\sqrt{\sigma})} \int\limits_1^{\sqrt{\sigma}} v\psi(v)dv = \infty.$$

Then the following asymptotic equality holds as $\sigma \to \infty$:

$$\mathcal{E}\left(\widehat{C}^{\psi}_{\beta,\infty}; W_\sigma\right)_{\widehat{C}} = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\sigma} \int\limits_1^{\sqrt{\sigma}} v\psi(v)dv + O\left(\psi(\sqrt{\sigma})\right). \tag{45}$$

Examples of functions that satisfy the conditions of Corollary 2 are the functions $\psi \in \mathfrak{A}^*$ that have the following form for $v \geq 1$:

$$\psi(v) = \frac{1}{v^2}\ln^\alpha(v+K), \quad K > 0, \quad \alpha > 0,$$

Corollary 3. *Suppose that* $\psi \in \mathfrak{A}_0^*$, $\sin\dfrac{\beta\pi}{2} \neq 0$, *the function* $v^2\psi(v)$ *is convex downwards on the interval* $[b,\infty)$, $b \geq 1$, *and*

$$\lim_{v\to\infty} v^2\psi(v) = K < \infty,$$

$$\lim_{\sigma\to\infty} \int\limits_1^{\sqrt{\sigma}} v\psi(v)dv = \infty.$$

Then the following asymptotic equality holds as $\sigma \to \infty$:

$$\mathcal{E}\left(\widehat{C}^{\psi}_{\beta,\infty}; W_\sigma\right)_{\widehat{C}} = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\sigma} \int\limits_1^{\sqrt{\sigma}} v\psi(v)dv + O\left(\frac{1}{\sigma}\right). \tag{46}$$

Examples of functions $\psi \in \mathfrak{A}^*$ that satisfy Corollary 3 are the functions that have the following form for $v \geq 1$:

$$\psi(v) = \frac{1}{v^2}(K + e^{-v}), \quad \psi(v) = \frac{1}{v^2\ln^\alpha(v+K)}, \quad K > 0, \quad 0 \leq \alpha \leq 1.$$

Note that, under the conditions of Corollaries 1–3, equalities (44)–(46) give a solution of the Kolmogorov–Nikol'skii problem for the Weierstrass operators W_σ on the classes $\widehat{C}^{\psi}_{\beta,\infty}$ in the uniform metric.

Let G^* be the set of functions $\psi \in \mathfrak{A}^*$ that satisfy the following condition: For an arbitrary constant $K > 0,$ there exists a point $v_0 = v_0(K) \geq 1$ such that, for $v > v_0,$ a function $\alpha(v)$ of the form (43) satisfies the inequality

$$\alpha(v) < \frac{1}{2}\left(1 - \frac{K}{v^2}\right).$$

Theorem 2. *Suppose that* $\psi \in G^*,$ *the function* $g(v) = v^2\psi(v)$ *is convex downwards on* $[b, \infty),$ $b \geq 1,$ *and*

$$\int\limits_1^\infty v\psi(v)dv < \infty. \tag{47}$$

Then the following asymptotic equality holds as $\sigma \to \infty:$

$$\mathcal{E}\left(\widehat{C}^\psi_{\beta,\infty}; W_\sigma\right)_{\widehat{C}} = \frac{1}{\sigma} \sup_{f \in \widehat{C}^\psi_{\beta,\infty}} \left\| f^{(2)}(x) \right\|_{\widehat{C}} + O\left(\frac{1}{\sigma\sqrt{\sigma}} \int\limits_1^{\sqrt{\sigma}} t^2\psi(t)dt + \frac{1}{\sigma} \int\limits_{\sqrt{\sigma}}^\infty t\psi(t)dt \right), \tag{48}$$

where $f^{(2)}(x)$ *is the second derivative of the function* $f(x)$.

Proof. We represent the function $\tau(v)$ defined by (5) in the form $\tau(v) = \varphi(v) + \mu(v),$ where

$$\varphi(v) = v^2 \frac{\psi(\sqrt{\sigma}v)}{\psi(\sqrt{\sigma})}, \quad v \geq 0, \tag{49}$$

$$\mu(v) = (1 - e^{-v^2} - v^2)\frac{\psi(\sqrt{\sigma}v)}{\psi(\sqrt{\sigma})}, \quad v \geq 0. \tag{50}$$

Let us verify the integrability of the transforms $\hat{\varphi}_\beta(t)$ and $\hat{\mu}_\beta(t)$ of the functions $\varphi(v)$ and $\mu(v)$ [see (9)].

We show the convergence of the integral

$$A(\varphi) = \frac{1}{\pi} \int\limits_{-\infty}^\infty \left| \int\limits_0^\infty \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt.$$

Integrating twice by parts and taking into account that

$$\varphi(0) = \varphi'(0) = 0 \qquad \text{and} \qquad \lim_{v\to\infty} \varphi(v) = \lim_{v\to\infty} \varphi'(v) = 0,$$

we obtain

$$\int_0^{\infty} \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

$$= \left(\int_0^{\frac{1}{\sqrt{\sigma}}} + \int_{\frac{1}{\sqrt{\sigma}}}^{\infty} \right) \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

$$= -\frac{1}{t^2} \left(\int_0^{\frac{1}{\sqrt{\sigma}}} + \int_{\frac{1}{\sqrt{\sigma}}}^{\infty} \right) \varphi''(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv + \frac{1}{t^2} \frac{\psi'(1-0) - \psi'(1+0)}{\sqrt{\sigma}\psi(\sqrt{\sigma})} \cos\left(\frac{1}{\sqrt{\sigma}} t + \frac{\beta\pi}{2}\right),$$

whence

$$\left| \int_0^{\infty} \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| \le \frac{1}{t^2} \left(\int_0^{\frac{1}{\sqrt{\sigma}}} + \int_{\frac{1}{\sqrt{\sigma}}}^{\infty} \right) |\varphi''(v)| dv + \frac{1}{t^2} \frac{K}{\sqrt{\sigma}\psi(\sqrt{\sigma})}.$$

The function $\varphi(v)$ is convex downwards on each of the intervals $\left[0, \frac{1}{\sqrt{\sigma}}\right)$ and $\left[\frac{b}{\sqrt{\sigma}}, \infty\right)$ and is bounded on the interval $\left(\frac{1}{\sqrt{\sigma}}, \frac{b}{\sqrt{\sigma}}\right]$. Using the last inequality, we obtain

$$\left| \int_0^{\infty} \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| \le \frac{1}{t^2} \left(\int_0^{\frac{1}{\sqrt{\sigma}}} + \int_{\frac{1}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}} + \int_{\frac{b}{\sqrt{\sigma}}}^{\infty} \right) |\varphi''(v)| dv + \frac{1}{t^2} \frac{K}{\sqrt{\sigma}\psi(\sqrt{\sigma})}$$

$$= \frac{1}{t^2} \left(\int_0^{\frac{1}{\sqrt{\sigma}}} + \int_{\frac{b}{\sqrt{\sigma}}}^{\infty} \right) \varphi''(v)\, dv + \frac{1}{t^2} \int_{\frac{1}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}} |\varphi''(v)|\, dv + \frac{1}{t^2} \frac{K}{\sqrt{\sigma}\psi(\sqrt{\sigma})}$$

$$\le \frac{K_1}{t^2\sqrt{\sigma}\psi(\sqrt{\sigma})} + \frac{1}{t^2\sqrt{\sigma}\psi(\sqrt{\sigma})} \int_1^b \left(2\psi(v) + 4v\psi'(v) + v^2\psi''(v)\right) dv$$

$$\le \frac{K_2}{t^2\sqrt{\sigma}\psi(\sqrt{\sigma})}.$$

Then

$$\int_{|t| \ge \sqrt{\sigma}} \left| \int_0^{\infty} \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt = O\left(\frac{1}{\sigma\psi(\sqrt{\sigma})}\right), \quad \sigma \to \infty. \tag{51}$$

Since the function $v^2\psi(v)$ decreases on $[b,\infty)$ and is bounded on $[1,b]$ and the function $\psi(v)$ increases on $[0,1],$ using relation (49) and equality (4.16) from [14, p. 59] we get

$$\int\limits_0^{\sqrt{\sigma}}\left|\int\limits_0^{\infty}\varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt=\int\limits_0^{\sqrt{\sigma}}\left|\left(\int\limits_0^{\frac{1}{\sqrt{\sigma}}}+\int\limits_{\frac{1}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}}+\int\limits_{\frac{b}{\sqrt{\sigma}}}^{\infty}\right)\varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt$$

$$\leq\sqrt{\sigma}\left(\int\limits_0^{\frac{1}{\sqrt{\sigma}}}+\int\limits_{\frac{1}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}}\right)|\varphi(v)|\,dv+\int\limits_0^{\sqrt{\sigma}}\int\limits_{\frac{b}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}+\frac{2\pi}{t}}\frac{v^2\psi(\sqrt{\sigma}v)}{\psi(\sqrt{\sigma})}dvdt$$

$$\leq\frac{\sqrt{\sigma}\psi(1)}{\psi(\sqrt{\sigma})}\int\limits_0^{\frac{1}{\sqrt{\sigma}}}v^2dv+\frac{1}{\sigma\psi(\sqrt{\sigma})}\int\limits_1^{b}v^2\psi(v)\,dv$$

$$+\frac{1}{\psi(\sqrt{\sigma})}\int\limits_0^{\sqrt{\sigma}}\int\limits_{\frac{b}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}+\frac{2\pi}{t}}v^2\psi(\sqrt{\sigma}v)dvdt$$

$$\leq\frac{K}{\sigma\psi(\sqrt{\sigma})}+\frac{1}{\psi(\sqrt{\sigma})}\int\limits_0^{\sqrt{\sigma}}\int\limits_{\frac{b}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}+\frac{2\pi}{t}}v^2\psi(\sqrt{\sigma}v)dvdt. \tag{52}$$

Taking into account equalities (66)–(70) from [11], we can write the following estimate:

$$\frac{1}{\psi(\sqrt{\sigma})}\int\limits_0^{\sqrt{\sigma}}\int\limits_{\frac{b}{\sqrt{\sigma}}}^{\frac{b}{\sqrt{\sigma}}+\frac{2\pi}{t}}v^2\psi(\sqrt{\sigma}v)dvdt=O\left(\frac{1}{\sigma\psi(\sqrt{\sigma})}\right). \tag{53}$$

Combining (52) and (53), we get

$$\int\limits_0^{\sqrt{\sigma}}\left|\int\limits_0^{\infty}\varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt=O\left(\frac{1}{\sigma\psi(\sqrt{\sigma})}\right),\quad \sigma\to\infty. \tag{54}$$

By analogy, one can show that

$$\int\limits_{-\sqrt{\sigma}}^{0}\left|\int\limits_0^{\infty}\varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt=O\left(\frac{1}{\sigma\psi(\sqrt{\sigma})}\right),\quad \sigma\to\infty. \tag{55}$$

Using relations (51), (54), and (55), we obtain

$$A(\varphi) = O\left(\frac{1}{\sigma\psi(\sqrt{\sigma})}\right), \quad \sigma \to \infty.$$

We now show the convergence of the integral

$$A(\mu) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \left| \int\limits_{0}^{\infty} \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt.$$

Integrating twice by parts and taking into account that

$$\mu(0) = \mu'(0) = 0 \qquad \text{and} \qquad \lim_{v\to\infty} \mu(v) = \lim_{v\to\infty} \mu'(v) = 0,$$

we get

$$\int\limits_{0}^{\infty} \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv = \left(\int\limits_{0}^{\frac{1}{\sqrt{\sigma}}} + \int\limits_{\frac{1}{\sqrt{\sigma}}}^{\infty} \right) \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

$$= -\frac{1}{t^2} \left(\int\limits_{0}^{\frac{1}{\sqrt{\sigma}}} + \int\limits_{\frac{1}{\sqrt{\sigma}}}^{\infty} \right) \mu''(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

$$- \frac{1}{t^2} \left(1 - e^{-\frac{1}{\sigma}} - \frac{1}{\sigma}\right) \frac{\sqrt{\sigma}\,(\psi'(1-0) - \psi'(1+0))}{\psi(\sqrt{\sigma})} \cos\left(\frac{t}{\sqrt{\sigma}} + \frac{\beta\pi}{2}\right),$$

whence

$$\left| \int\limits_{0}^{\infty} \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| \le \frac{1}{t^2} \left(\int\limits_{0}^{\frac{1}{\sqrt{\sigma}}} + \int\limits_{\frac{1}{\sqrt{\sigma}}}^{\infty} \right) \left|\mu''(v)\right| dv + \frac{1}{t^2} \frac{K}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})}. \tag{56}$$

Let us estimate the integrals on the right-hand side of inequality (56). Taking into account that $\mu''(v) < 0$, $v \in \left[0, \dfrac{1}{\sqrt{\sigma}}\right]$, we obtain

$$\int\limits_{0}^{\frac{1}{\sqrt{\sigma}}} \left|\mu''(v)\right| dv = -\int\limits_{0}^{\frac{1}{\sqrt{\sigma}}} \mu''(v) dv = \frac{2\psi(1)}{\sqrt{\sigma}\psi(\sqrt{\sigma})} \left(1 - e^{-\frac{1}{\sigma}}\right) - \frac{\sqrt{\sigma}\psi'(1-0)}{\psi(\sqrt{\sigma})} \left(1 - e^{-\frac{1}{\sigma}} - \frac{1}{\sigma}\right)$$

$$\le \frac{K}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})}. \tag{57}$$

According to inequality (79) in [11], the following estimate holds for the second integral on the right-hand side of inequality (56):

$$\int_{\frac{1}{\sqrt{\sigma}}}^{\infty} \left|\mu''(v)\right| dv \le K + \frac{K_1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})} + \frac{K_2}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})} \int_1^{\sqrt{\sigma}} v^2\psi(v)dv. \tag{58}$$

Using (56)–(58) and the relation

$$\lim_{\sigma\to\infty} \frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})} \int_1^{\sqrt{\sigma}} v^2\psi(v)dv \ge \lim_{\sigma\to\infty} \frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})}\sigma\psi(\sqrt{\sigma}) \int_1^{\sqrt{\sigma}} dv = 1, \tag{59}$$

we obtain

$$\int_{|t|\ge\pi} \left|\int_0^{\infty} \mu(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\right| dt = O\left(\frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})} \int_1^{\sqrt{\sigma}} v^2\psi(v)dv\right). \tag{60}$$

Now consider

$$\int_0^{\pi} \left|\int_0^{\infty} \mu(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\right| dt \le \int_0^{\pi} \left|\left(\int_0^{\frac{1}{\sqrt{\delta}}} + \int_{\frac{1}{\sqrt{\delta}}}^{1} + \int_1^{\infty}\right) \mu(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\right| dt. \tag{61}$$

Taking into account that the function $\psi(v)$ increases on $[0,1]$ and using the inequality

$$e^{-v^2} + v^2 - 1 \le v^2, \quad v \in R, \tag{62}$$

we obtain

$$\int_0^{\pi} \left|\int_0^{\frac{1}{\sqrt{\sigma}}} \mu(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\right| dt \le \int_0^{\pi}\int_0^{\frac{1}{\sqrt{\sigma}}} |\mu(v)|\, dvdt \le \frac{\pi\psi(1)}{\psi(\sqrt{\sigma})} \int_0^{\frac{1}{\sqrt{\sigma}}} \left(e^{-v^2} + v^2 - 1\right) dv$$

$$\le \frac{\pi\psi(1)}{\psi(\sqrt{\sigma})} \int_0^{\frac{1}{\sqrt{\sigma}}} v^2 dv \le \frac{K}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})}. \tag{63}$$

According to relation (85) in [11], the following estimate is true:

$$\int_0^{\pi} \left|\int_{\frac{1}{\sqrt{\sigma}}}^{1} \mu(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\right| dt = O\left(\frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})} \int_1^{\sqrt{\sigma}} v^2\psi(v)dv\right). \tag{64}$$

Since $\psi \in G^*$, it is easy to verify that the function $-\mu(v) = (e^{-v^2} + v^2 - 1)\psi(\sqrt{\sigma}v)$ monotonically decreases beginning with a certain value $v_1 \geq 1$.

In view of the fact that the function $-\mu(v)$ monotonically decreases on $[v_1, \infty)$, $v_1 \geq 1$, is nonnegative, and vanishes as $v \to \infty$, we can use inequality (4.16) from [14, p. 59]. Using also inequality (62), we get

$$\int_0^\pi \left| \int_1^\infty \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt = \int_0^\pi \left| \int_1^\infty (-\mu(v)) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt$$

$$\leq \int_0^\pi \left| \int_1^{v_1} (-\mu(v)) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt$$

$$+ \int_0^\pi \left| \int_{v_1}^\infty (-\mu(v)) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt$$

$$\leq \int_0^\pi \int_1^{v_1} (-\mu(v))\, dvdt + \int_0^\pi \int_{v_1}^{v_1 + \frac{2\pi}{t}} (-\mu(v))\, dvdt$$

$$= \int_0^\pi \int_1^{v_1 + \frac{2\pi}{t}} (-\mu(v))\, dvdt \leq \frac{1}{\psi(\sqrt{\sigma})} \int_0^\pi \int_1^{v_1 + \frac{2\pi}{t}} v^2 \psi(\sqrt{\sigma}v) dvdt. \tag{65}$$

According to relation (93) in [11], we obtain the following estimate for the last integral in (65):

$$\frac{1}{\psi(\sqrt{\sigma})} \int_0^\pi \int_1^{v_1 + \frac{2\pi}{t}} v^2 \psi(\sqrt{\sigma}v) dvdt = O\left(1 + \frac{1}{\sigma\psi(\sqrt{\sigma})} \int_{\sqrt{\sigma}}^\infty v\psi(v) dv\right). \tag{66}$$

Combining relations (65) and (66), we get

$$\int_0^\pi \left| \int_1^\infty \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt = O\left(1 + \frac{1}{\sigma\psi(\sqrt{\sigma})} \int_{\sqrt{\sigma}}^\infty v\psi(v) dv\right). \tag{67}$$

Using (63), (64), (67), and (59), we deduce the following relation from (61):

$$\int_0^\pi \left| \int_0^\infty \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt = O\left(\frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})} \int_1^{\sqrt{\sigma}} v^2 \psi(v) dv + \frac{1}{\sigma\psi(\sqrt{\sigma})} \int_{\sqrt{\sigma}}^\infty v\psi(v) dv\right). \tag{68}$$

By analogy, one can show that the following estimate is true:

$$\int\limits_{-\pi}^{0}\left|\int\limits_{0}^{\infty}\mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt = O\left(\frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})}\int\limits_{1}^{\sqrt{\sigma}}v^2\psi(v)dv+\frac{1}{\sigma\psi(\sqrt{\sigma})}\int\limits_{\sqrt{\sigma}}^{\infty}v\psi(v)dv\right). \quad (69)$$

Combining relations (60), (68), and (69), we get

$$A\left(\mu\right) = O\left(\frac{1}{\sigma\sqrt{\sigma}\psi(\sqrt{\sigma})}\int\limits_{1}^{\sqrt{\sigma}}v^2\psi(v)dv+\frac{1}{\sigma\psi(\sqrt{\sigma})}\int\limits_{\sqrt{\sigma}}^{\infty}v\psi(v)dv\right). \quad (70)$$

Taking (5) into account, we deduce the following equality from (2) and (4):

$$f(x)-W_\sigma\left(f,x\right) = \psi(\sqrt{\sigma})\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sqrt{\sigma}}\right)\frac{1}{\pi}\int\limits_{0}^{\infty}\tau(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\,dt. \quad (71)$$

Then

$$\begin{aligned}
\mathcal{E}\left(\widehat{C}^\psi_{\beta,\infty};W_\sigma\right)_{\widehat{C}} &= \sup_{f\in\widehat{C}^\psi_{\beta,\infty}}\|f(x)-W_\sigma(f;x)\|_{\widehat{C}}\\
&= \sup_{f\in\widehat{C}^\psi_{\beta,\infty}}\left\|\psi(\sqrt{\sigma})\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sqrt{\sigma}}\right)\frac{1}{\pi}\int\limits_{0}^{\infty}\tau(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dvdt\right\|_{\widehat{C}}\\
&= \sup_{f\in\widehat{C}^\psi_{\beta,\infty}}\left\|\psi(\sqrt{\sigma})\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sqrt{\sigma}}\right)\widehat{\tau}_\beta(t)dt\right\|_{\widehat{C}}\\
&= \sup_{f\in\widehat{C}^\psi_{\beta,\infty}}\left\|\psi(\sqrt{\sigma})\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sqrt{\sigma}}\right)\left(\widehat{\varphi}_\beta(t)+\widehat{\mu}_\beta(t)\right)dt\right\|_{\widehat{C}}\\
&= \sup_{f\in\widehat{C}^\psi_{\beta,\infty}}\left\|\psi(\sqrt{\sigma})\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sqrt{\sigma}}\right)\widehat{\varphi}_\beta(t)dt\right\|_{\widehat{C}}+O\left(\psi(\sqrt{\sigma})A(\mu)\right). \quad (72)
\end{aligned}$$

It is easy to verify that

$$\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sqrt{\sigma}}\right)\widehat{\varphi}_\beta(t)dt = \frac{1}{\sigma\psi(\sqrt{\sigma})}\frac{1}{\pi}\int\limits_{-\infty}^{\infty}f_\beta^\psi(x+t)\int\limits_{0}^{\infty}v^2\psi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\,dt = -f^{(2)}(x), \quad (73)$$

where $f^{(2)}(x)$ is the second derivative of the function $f(x)$.

Substituting (73) into (72), we obtain

$$\mathcal{E}\left(\widehat{C}^{\psi}_{\beta,\infty}; W_\sigma\right)_{\widehat{C}} = \frac{1}{\sigma} \sup_{f \in \widehat{C}^{\psi}_{\beta,\infty}} \left\| f^{(2)}(x) \right\|_{\widehat{C}} + O\left(\psi(\sqrt{\sigma})A(\mu)\right), \quad \sigma \to \infty. \tag{74}$$

Equality (48) follows from (74) and (70).
Theorem 2 is proved.

Examples of functions for which Theorem 2 is true are the functions $\psi \in \mathfrak{A}^*$ that have the following form on $[1, \infty)$:

$$\psi(v) = \frac{1}{v^2 \ln^{\alpha}(v + K)}, \quad K > 0, \quad \alpha > 1,$$

$$\psi(v) = \frac{1}{v^r} \ln^{\alpha}(v + K), \quad \psi(v) = \frac{1}{v^r} \arctan v, \quad \psi(v) = \frac{1}{v^r}(K + e^{-v}), \quad K > 0, \quad r > 2, \quad \alpha \in R.$$

3. Estimation of Upper Bounds of Functions from the Class $\widehat{L}^{\psi}_{\beta,1}$ by Weierstrass Operators in the Integral Metric

By virtue of the lemma in [15] and Lemma 1 in [12], the following equality holds for the function $\tau(v)$ defined by (5):

$$\mathcal{E}\left(\widehat{L}^{\psi}_{\beta,1}; W_\sigma\right)_{\widehat{1}} = \mathcal{E}\left(\widehat{C}^{\psi}_{\beta,1}; W_\sigma\right)_{\widehat{C}} + O\left(\psi(\sqrt{\sigma})\gamma(\sigma)\right), \quad \sigma \to \infty,$$

where $\gamma(\sigma) \le 0$ and

$$|\gamma(\sigma)| = O\left(\int\limits_{|t| \ge \frac{\sqrt{\sigma}\pi}{2}} \left| \int\limits_0^{\infty} \tau(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt \right). \tag{75}$$

The following theorem is true:

Theorem 3. *Suppose that* $\psi \in \mathfrak{A}^*_0 \cap \mathfrak{A}'$ *and the function* $g(v) = v^2\psi(v)$ *is convex upwards or downwards on* $[b, \infty)$, $b \ge 1$. *Then the following equality holds as* $\sigma \to \infty$:

$$\mathcal{E}\left(\widehat{L}^{\psi}_{\beta,1}; W_\sigma\right)_{\widehat{1}} = \psi(\sqrt{\sigma})A(\tau) + O\left(\frac{1}{\sigma} + \frac{\psi(\sqrt{\sigma})}{\sqrt{\sigma}}\right),$$

where $A(\tau)$ *is defined by equality (7) and satisfies estimate (8).*

Theorem 3 follows from Theorem 1 and the estimate (see (48) in [11])

$$\int\limits_{|t| \ge \frac{\sqrt{\sigma}\pi}{2}} \left| \int\limits_0^{\infty} \tau(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt = O\left(\frac{1}{\sigma\psi(\sqrt{\sigma})} + \frac{1}{\sqrt{\sigma}}\right), \quad \sigma \to \infty.$$

Corollary 4. *If the conditions of Theorem 1 are satisfied,* $\sin\dfrac{\beta\pi}{2}\neq 0,$ *and* $\lim\limits_{t\to\infty}\alpha(t)=\infty,$ *where* $\alpha(t)$ *is defined by (43), then the following asymptotic equality holds as* $\sigma\to\infty:$

$$\mathcal{E}\left(\widehat{L}^{\psi}_{\beta,1};W_{\sigma}\right)_{\widehat{1}}=\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int\limits_{\sqrt{\sigma}}^{\infty}\frac{\psi(v)}{v}dv+O\left(\psi(\sqrt{\sigma})\right). \tag{76}$$

Corollary 5. *Suppose that* $\psi\in\mathfrak{A}_0^*,$ $\sin\dfrac{\beta\pi}{2}\neq 0,$ *the function* $v^2\psi(v)$ *is convex upwards or downwards on* $[b,\infty),$ $b\geq 1,$ *and*

$$\lim_{v\to\infty}v^2\psi(v)=\infty,$$

$$\lim_{\sigma\to\infty}\frac{1}{\sigma\psi(\sqrt{\sigma})}\int\limits_{1}^{\sqrt{\sigma}}v\psi(v)dv=\infty.$$

Then the following asymptotic equality holds as $\sigma\to\infty:$

$$\mathcal{E}\left(\widehat{L}^{\psi}_{\beta,1};W_{\sigma}\right)_{\widehat{1}}=\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\frac{1}{\sigma}\int\limits_{1}^{\sqrt{\sigma}}v\psi(v)dv+O\left(\psi(\sqrt{\sigma})\right). \tag{77}$$

Corollary 6. *Suppose that* $\psi\in\mathfrak{A}_0^*,$ $\sin\dfrac{\beta\pi}{2}\neq 0,$ *the function* $v^2\psi(v)$ *is convex downwards on* $[b,\infty),$ $b\geq 1,$ *and*

$$\lim_{v\to\infty}v^2\psi(v)=K<\infty,$$

$$\lim_{\sigma\to\infty}\int\limits_{1}^{\sqrt{\sigma}}v\psi(v)dv=\infty.$$

Then the following asymptotic equality holds as $\sigma\to\infty:$

$$\mathcal{E}\left(\widehat{L}^{\psi}_{\beta,1};W_{\sigma}\right)_{\widehat{1}}=\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\frac{1}{\sigma}\int\limits_{1}^{\sqrt{\sigma}}v\psi(v)dv+O\left(\frac{1}{\sigma}\right). \tag{78}$$

Note that, under the conditions of Corollaries 4–6, equalities (76)–(78) give a solution of the Kolmogorov–Nikol'skii problem for the Weierstrass operators W_σ on the classes $\widehat{L}^{\psi}_{\beta,1}$ in the integral metric.

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